$1 + 2 + 3 + \ldots \neq -\frac{1}{12}$ An Introduction to Analytic Continuation

Matt DeCross

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Functions are often defined by series that don't converge everywhere:

$$f(x) = 1 + x + x^2 + x^3 + \dots \qquad |x| < 1$$

Sometimes, there exists a different representation of the function which is equal to the series where it converges:

$$\begin{aligned} f(x) &= 1 + x + x^2 + x^3 + \dots \\ &= \frac{1}{1-x} \qquad |x| < 1 \end{aligned}$$

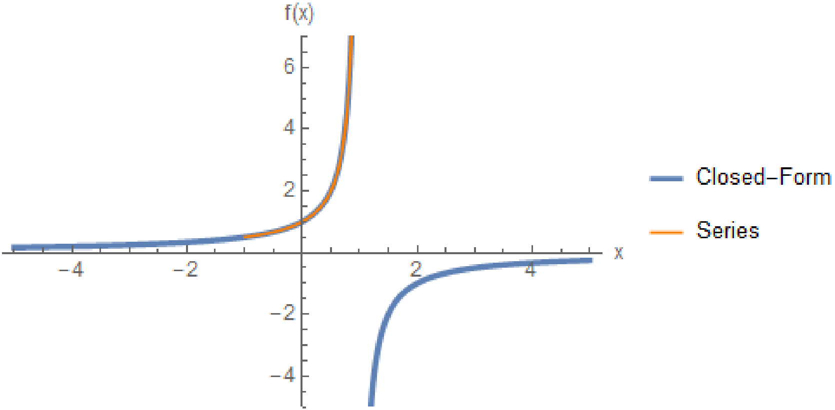
The new representation gives reasonable values even outside the domain of convergence:

$$f(x) = \frac{1}{1-x} \implies f(3) = -\frac{1}{2}$$

But

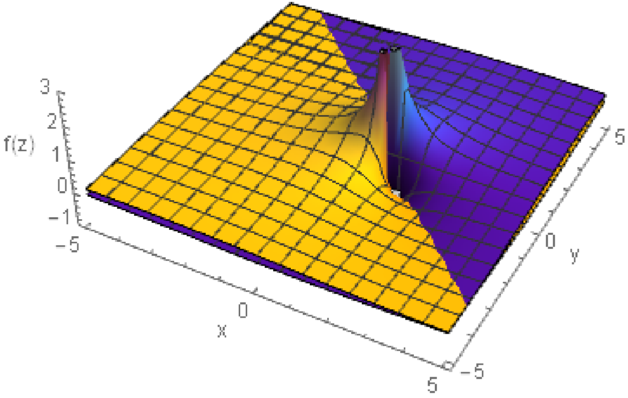
$$1 + 3 + 3^2 + 3^3 + \ldots \to \infty$$

Both the series and the closed-form still blow up at some of the same points:



The closed-form even works for complex numbers,

$$f(z) = \frac{1}{1-z} = \frac{1}{1-x-iy}$$



So what do we really mean by equals?

Does $f(x) = 1 + x + x^2 + x^3 + \ldots$ or does $f(x) = \frac{1}{1-x}$? Clearly they disagree sometimes.

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All other representations (like series with a finite radius of convergence) just happen to agree with f where they converge.

Conclusion: $f(z) = \frac{1}{1-z}$. The function f has been extended over the complex numbers.

Analytic continuation: The extension of a function* to a larger domain

* Function must be "analytic" $\sim$ "nice enough."

How do we magically extend things in practice? Won't always get lucky with a closed-form for the series.

Answer: usually by noticing patterns!

Example: the *Gamma function* $\Gamma(z)$ and the *factorial* $n! = n(n-1)(n-2)\dots 1$. How do we define the factorial for real numbers x that are not integers? Or for complex numbers z?

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Notice the factorial is defined recursively by $0! = 1$ and $n! = n(n-1)!$

Motivates definition of an “analytic” function $\Gamma(z)$, with $\Gamma(1) = 1$ and $\Gamma(z + 1) = z\Gamma(z)$.

When z is a positive integer, $\Gamma(z) = (z - 1)!$

How to write down a function from this? Easiest way is to be playing with integrals and just notice that the following satisfies the same recursion relations by integrating by parts:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

Lots of complex analysis machinery also works, to represent it as an infinite product of functions.

$\Gamma(z)$ is not the factorial. What is $(2.5)!$? What is $(3+2i)!$? But $\Gamma(3) = 2!$. $\Gamma(z)$ is the analytic continuation of the factorial.

Player 2 enters the game: the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

The sum does not converge when $s \leq 1$, if s is a real number.

However, the zeta function also satisfies the following relation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

How to prove this? It's constructed magically from infinite product representations with the help of the gamma function.

Notice that

$$\begin{aligned}\zeta(-1) &= 2^{-1}\pi^{-2}\sin\left(-\frac{\pi}{2}\right)\Gamma(2)\zeta(2)\\ &= \frac{1}{2\pi^2}(-1)(1!)\frac{\pi^2}{6}\\ &= -\frac{1}{12}\end{aligned}$$

The fact that $\zeta(2) = \frac{\pi^2}{6}$ is well-known and cute but tricky to prove. The easiest way uses Fourier series, which are in most differential equations classes.

In summary we have found that $\zeta(-1) = -\frac{1}{12}$.

And if we plug $s = -1$ into the series, it looks like

$$1 + 2 + 3 + \dots$$

But $1 + 2 + 3 + \dots \neq -\frac{1}{12}$! $\zeta(-1) \neq 1 + 2 + 3 + \dots$! It is the analytic continuation that we evaluate at $s = -1$, not the series.

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This trick works to assign finite values to other series as well, like $\zeta(0) = 1 + 1 + 1 + \ldots = -\frac{1}{2}$.

In physics: the sum $1 + 2 + 3 + \ldots$ shows up frequently when computing the energy of quantum fluctuations of the vacuum.

Two motivations for replacing the series with its continuation:

- Experimental observation (e.g. Casimir effect)
- Alternate methods of computation (nilpotency of BRST charge, vanishing of conformal anomaly vs. Lorentz anomaly fixing spacetime dimension in (super)string theory).

$1 + 2 + 3 + \ldots \neq -\frac{1}{12}$, but the replacement is often convenient.

Questions?

References:

Lars Ahlfors. *Complex Analysis*. McGraw-Hill, 1980.