Nakahara Ch. 2 Problems

Matt DeCross

June 14, 2016

Topics covered: Maps; Equivalence relations and classes; Orientability; Projective spaces; Elementary group theory; Vector spaces; Tensors; Topological spaces and elementary topology; Continuity; Compactness; Connectedness, Homeomorphism and topological invariants.

These problems remain challenging (I think reasonably so); but I have separated out a bit more math practice since this chapter is full of fundamental material. I tried to pull more interesting exercises rather than rote ones, although I think having easy problems is nice. I'm a little rusty on Aharonov-Bohm, so it would be nice if somebody could sharpen up my statement on 3e) – I know what I have in mind, but I think as is the problem just suggests at it.

1 Short Math Questions

(a) Nakahara Exercise 2.13: Suppose that the axioms of a topological space are modified so that infinite (not just finite) intersections of open sets are open. Show that this reduces the usual topology on $\mathbb{R}$ to the discrete topology.

(b) Show that adding the point at infinity $\{\infty\}$ to the Euclidean plane $\mathbb{R}^2$ is equivalent to compactifying the space to the sphere S^2 by explicitly constructing a homeomorphism between the two.

(c) How many equivalence relations can be defined on a set of six elements? (Source: modification of Problem 7.6 from Chapter 2 of Artin's *Algebra*). Note that equivalence relations are considered unique only if they define different sets of equivalence classes.

(d) Show that attaching $\mathbb{R}P^2$ to a surface by pipe decreases the Euler characteristic of the surface by one. Source: Course entitled Topological terms in condensed matter physics, Spring 2009, Problem Set 2, http://felix.physics.sunysb.edu/ abanov/Teaching/Spring2009/phy680.html.

(e) A **path** from a to b in a generic space X is a continuous function f from the interval $[0, 1]$ into X such that $f(0) = a$ and $f(1) = b$. In this problem, let G be an arbitrary subgroup of $GL_n(\mathbb{R})$, the set of real, invertible $n \times n$ matrices.

   (i) Prove that if $a, b, c, d \in G$, and there are paths in G from a to b and c to d, then there is a path in G from $a \cdot c$ to $b \cdot d$.

   (ii) The set of matrices that are path-connected to the identity in G is called the **connected component** of G. Prove that this set forms a normal subgroup of G.

Source: modification of Problem M.7 from Chapter 2 of Artin's *Algebra*.

2 Discrete Normal Subgroups

Source: Problem 1, Problem Set 4 from 18.755 (Introduction to Lie Groups), Fall 2014.

Recall that a *group* is a set closed under some associative law of composition ("multiplication") in which an identity and inverse element exist. A **Lie group** is one in which multiplication is continuous (in fact, it is smooth). Here we also define the **center** of a group as set of all elements of a group which commute with every element of the group.

Suppose $N \subset G$ is a normal subgroup of a connected Lie group G, and suppose the subspace topology on N is the discrete topology. Prove that N is contained in the center of G.

Note: you will need the fact that continuous maps preserve connectedness.

3 Background Independence: A "Topological Theta Term"

Source: Course entitled Topological terms in condensed matter physics, Spring 2009, Problem Set 1, http://felix.physics.sunysb.edu/ abanov/Teaching/Spring2009/phy680.html.

In condensed matter physics and string theory, a certain type of field theory is called a "topological quantum field theory" if the action and resulting spectrum of states is background-independent, meaning (in part) that it does not depend on specifying a metric in advance. This is very interesting and well-studied in the context of quantum gravity (particularly by Witten), since it would be useful to have an action for gravity that doesn't rely on having a metric in advance (which results in nonlinearity). In this problem we'll examine the sense in which background-independence is "topological" in a simple example. This will involve adding a term to the Lagrangian which is a topological invariant, dependent on the **fundamental group** of the space (a concept that will come up in later chapters).

Consider the classical action of a particle on a ring, with θ a constant and ϕ the angle on the ring:

$$S = \int d\tau \left(\frac{1}{2} m\dot{\phi}^2 - \frac{\theta}{2\pi}\dot{\phi} \right) \tag{3.1}$$

where τ is some "proper" time. Reparametrizing time as $\tau = f(t)$ we have $d\tau = f' dt$ and $d\tau^2 = (f')^2 dt^2$; identify a "metric" via $g_{00} = (f')^2$ and $g^{00} = (f')^{-2}$. From this we have $\sqrt{g_{00}} = f'$

(a) Rewrite the action in terms of $\phi(t)$ instead of $\phi(\tau)$.

(b) Find the form of the action if it is written in terms of the introduced metric (Note: the metric is often called an *einbein* when introduced as an auxiliary field like this).

(c) Using the general formula for the variation of the action with respect to a metric ($g = \det g_{\mu\nu}$):

$$\delta S = \int dx \sqrt{g} \frac{1}{2} T_{\mu\nu} \delta g^{\mu\nu}, \tag{3.2}$$

find the stress-energy tensor for the particle on the ring and confirm that T_{00} gives the energy of the particle. What is the contribution of the "topological theta term" to the stress-energy tensor?

(d) Suppose the particle travels around the ring once. By how much does the action increase/decrease?

(e) Secretly we've been looking at the quantum theory of electrons in a magnetic field! Recall from the last problem set that the Lagrangian of an electron in a magnetic field is:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + e\dot{x} \cdot A \tag{3.3}$$

Compare to the "topological" action in this problem. What does this imply about cyclic paths of an electron around a solenoid? We will revisit such a scenario later in discussion of the Aharonov-Bohm effect.

Thanks to Bob Knighton for the following addition to this problem:

To make the previous discussion more explicit: consider the following classical action of a charged particle moving in an electromagnetic field with vector potential $\mathbf{A}$ and no electric potential (ie $\phi = 0$)

$$S = \int \mathrm{d}t \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - e\dot{\mathbf{x}} \cdot \mathbf{A}\right)$$

f) Rewrite S in the simple case of a particle confined to move on a ring of radius R.

g) Let us apply this to the specific case where the electromagnetic field is generated by a solenoid of radius $a < R$ passing through the axis of rotation of the particle. The vector potential in a specific gauge is given by

$$\mathbf{A} = \begin{cases} (\Phi r/2\pi a^2)\,\hat{\boldsymbol{\phi}}, & r < a \\ (\Phi/2\pi r)\,\hat{\boldsymbol{\phi}}, & r > a \end{cases}$$

Where Φ is the magnetic flux passing through the solenoid. Use this to identify S with the free action of a particle on a ring plus the topological θ term given in the previous part. In particular, give an expression for θ in terms of given physical quantities.

h) The path integral formulation of quantum mechanics tells us that the amplitude to propogate from the point $\mathbf{x}_i$ at time t_i to the point $\mathbf{x}_f$ at time t_f is given by

$$\langle \mathbf{x}_f, t_f | \mathbf{x}_i, t_i \rangle = \int \mathcal{D}[\mathbf{x}(t)] \exp\left(-\frac{i}{\hbar}S[\mathbf{x}(t)]\right)$$

Where the action is defined on paths begining and ending at our desired times and points and where $\int \mathcal{D}[\mathbf{x}(t)]$ defines an integration over all integrable paths between these points. Using this, show that if an electron travels once around the solenoid, it picks up a phase factor and identify the value of this phase.

This effect is the famous **Aharanov-Bohm Effect**, which shows that a particle can be influenced by an electromagnetic field while never actually passing through it. The phase picked up as the electron travels around the solenoid is known as a **Topological Phase** (because the solenoid behaves as a topological defect altering the physics of the problem) or a **Berry's Phase**. These will be discussed later in the book and in later problems.