Nakahara Ch. 3 Problems

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Topics covered: Abelian groups (free, finite cyclic); First Isomorphism Theorem; Simplexes, simplicial complexes, and polyhedra; Chain, cycle, and boundary groups; Homology groups; Homology and connectedness; Betti numbers and Euler-Poincaré theorem.

A lot of computation of homology groups on this set – it turns out it's pretty hard to find any pure physics content on homology which doesn't reference cohomology or homotopy more (which we haven't gotten to yet). My understanding is that when we start to talk about differential forms and cohomology, integration is basically defined on homology classes of curves, so it's useful to understand.

1 Homology Groups of Spheres and Balls

(a) Compute all homology groups of the circle S^1.

(b) Compute all homology groups of the sphere S^2 (Note that "sphere" refers only to the surface, whereas "ball" as below includes the interior).

(c) Compute all homology groups of the (generalized) sphere S^d.

(d) Compute all homology groups of the d-dimensional ball B^d (for instance, B^1 is the solid disc).

(e) Nakahara 3.1: The most general orientable two-dimensional surface is a 2-sphere with h handles and q holes. Compute the homology groups and the Euler characteristic of this surface.

(f) Nakahara 3.2: Consider a sphere with a hole and identify the edges of the hole as shown in Figure 3.13 in Nakahara (that is, with opposite edges identified in opposite directions). The surface obtained by this identification is simply the projective plane $\mathbb{R}P^2$. More generally, consider a sphere with q such 'crosscaps' and compute the homology groups and Euler characteristic of this surface.

2 Homology vs. Homotopy

On the 2-torus T^2, there are two one-cycles that are not boundaries that are not in the same homology class. On the torus, homologous 1-cycles can be continuously deformed into one another, and 1-cycles that are boundaries can be continuously contracted to a point. These are special features of a torus that make it a bad example, because homology does not have anything to do with continuous deformations or contractions *a priori* (which belong to *homotopy theory*, the subject of the next chapter. Note however that there is a connection between the two; look up the Hurewicz theorems).These problems are to emphasize this point.

(a) Find (i.e. draw) a 2-dimensional manifold with a 1-cycle on it that is a boundary but is not contractible to a point.

(b) Find a 2-dimensional manifold with two homologous 1-cycles that cannot be continuously deformed into one another.

Source: Modification of problem 3 from http://bohr.physics.berkeley.edu/classes/222/hws/hw2.pdf UC Berkeley Physics 222 "Geometry and Topology for Physicists."

3 Torsion Subgroups and Orientability

For this problem, suppose that simplicial homology is defined with coefficients in $\mathbb{Z}_2$ instead of $\mathbb{Z}$. Given this definition:

(a) Find all homology groups of S^1.

(b) Find all homology groups of the Mobius strip

(c) Find all homology groups of S^2.

(d) Find all homology groups of the Klein bottle.

(e) Find all homology groups of $\mathbb{R}P^2$.

(f) Prove or disprove: when simplicial homology is defined with $\mathbb{Z}_2$ coefficients, all torsion subgroups are always trivial.