Nakahara Ch. 4 Problems

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Topics covered: Fundamental groups and homotopy; Free groups and relations; Trees and edge loops; Commutator subgroups; Higher homotopy groups; Universal covering spaces; Applications to defects in condensed matter systems.

This week features a number of guest problems introducing widely varying applications of homotopy, which I hope will become a recurring theme in later weeks, although there is a lot of material to work through below. I have also included some less extensive exercises and a pure math problem that I really like.

1 Short Math Problems

This problem consists just of three exercises: the first and third are not too hard, but are pretty important for later material. The second is just a problem that I like, since commutator subgroups are not emphasized in Nakahara.

(a) Nakahara 4.1: show that the n-sphere S^n is a deformation retract of the punctured Euclidean space $\mathbb{R}^{n+1} - \{0\}$. Explicitly construct a retraction. Note: for $n = 2$, the punctured Euclidean space $\mathbb{R}^3 - \{0\}$ is the topological setting of pointlike defects in space like magnetic monopoles. Since S^2 is compact, it is extremely convenient to consider the topology (i.e., homotopy groups and more) of S^2 when discussing magnetic monopoles.

(b) Let $\mathcal{F}$ be the free group on x, y and let $\mathcal{R}$ be the smallest normal subgroup containing the commutator $xyx^{-1}y^{-1}$.

  (a) Show that $x^2y^2x^{-2}y^{-2}$ is in $\mathcal{R}$.

  (b) Prove that $\mathcal{R}$ is the commutator subgroup of $\mathcal{F}$, i.e. the smallest subgroup containing all commutators.

  Source: Artin, Chapter 7, Problem 10.10.

(c) Nakahra 4.3: Recall from previous discussion that the first homology group of a connected space is given by the abelianization of the fundamental group. It may or may not be surprising that this fact is no longer true for higher homotopy groups. For instance, the third (and higher) homology groups of S^2 are all trivial. But $\pi_3(S^2)$ is isomorphic to $\mathbb{Z}$.

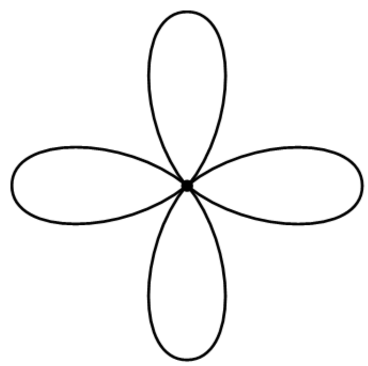
So, the problem: construct maps $f : S^3 \to S^2$ which belong to the elements 0 and 1 of $\pi_3(S^2)$. For more information as to why this problem is interesting and/or help solving it, look up the *Hopf fibration*.

2 Bouquets and Earrings: A Love Story

A few weeks ago there was some discussion over whether there was a difference between the direct sum and Cartesian product for Abelian groups. This problem will explore whether or not this is the case in the context of the homotopy and homology of two very interesting, similar-looking spaces.

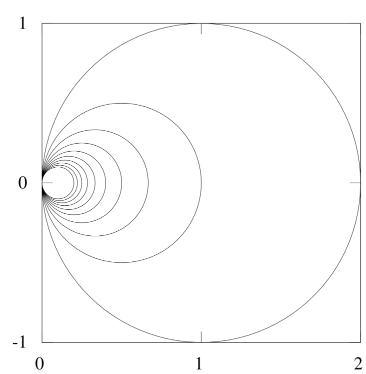
Consider the following two spaces:

The **n-bouquet** $\mathcal{B}_n$, also called the **rose of n petals**:



This space is formed by the so-called **wedge product** of n copies of S^1; where the wedge product is defined by attaching each copy of S^1 to the same point as shown above for the 4-bouquet. Note that the space $\mathcal{B}_n$ should be considered as having a topology coming from this product structure of copies of S^1 and *not* the subspace topology of $\mathbb{R}^2$ looking at this figure as embedded in the plane.

This is an important distinction to make, because here is another interesting space; the **Hawaiian earring** $\mathcal{E}_n$ of n rings:



This is the topological space given by the union of n circles attached at the origin in $\mathbb{R}^2$ with center $(1/k, 0)$ and radius $1/k$ for the kth circle.

(a) Find the fundamental groups of $\mathcal{B}_n$ and $\mathcal{E}_n$ for finite n.

(b) Find the first homology groups of $\mathcal{B}_n$ and $\mathcal{E}_n$ for finite n.

Okay, the groups agree for finite n. Definitely nothing bad will happen when we take $n \to \infty$, right?

(c) Find the fundamental group of $\mathcal{B}_\infty$.

(d) Explain why there is a natural injective map from $\pi_1(\mathcal{B}_\infty) \to \pi_1(E_\infty)$. Find an element of $\pi_1(\mathcal{E}_\infty)$ that is not in the image of this map. (*Thanks to Semon Rezchikov for pointing out to me that the fundamental group of $\mathcal{E}_\infty$ was more complicated than I initially thought*).

Although the fundamental groups seem like they should be similar, the above suggests that the fundamental groups are not the same; the fundamental group of $\mathcal{E}_\infty$ is larger. But the distinction in the previous part is pretty minor, and the spaces look very much alike, so it's not really that convincing that this means there is no homeomorphism between them. The following should convince you of this, however incredible it seems:

(e) Show that $\mathcal{B}_\infty$ and $\mathcal{E}_\infty$ are not homeomorphic by showing that $\mathcal{B}_\infty$ is not compact while $\mathcal{E}_\infty$ is. Recall that compactness means that for every open cover (collection of open sets covering a space) there exists a finite subcover.

Okay, I claim this has something to do with the infinite direct sum vs. infinite Cartesian product. What is that?

(f) Compute the first homology group of $\mathcal{B}_\infty$.

(g) The first homology group of $\mathcal{E}_\infty$ can be written as $\mathcal{J} \times \prod_{n=1}^{\infty} \mathbb{Z}$, where $\mathcal{J}$ is a terrible factor that I won't pretend to understand that involves p-adic numbers – if interested, see http://jlms.oxfordjournals.org/content/62/1/305 (this might seem confusing, if you think about the Abelianization of a word of infinite length in $\pi_1(\mathcal{E}_\infty)$ – I believe the problem comes from the fact that it takes infinite steps to reduce such a word, which results in the Abelianization not being just the naïve reduction into the product of $\mathbb{Z}$ generators to some powers). Ignoring this factor of $\mathcal{J}$, then, the first homology group of $\mathcal{E}_\infty$ is the countably direct product of copies of $\mathbb{Z}$. Explain the connection of the following fact about the difference between infinite direct sum and infinite Cartesian product to the context of this problem: the countably infinite direct sum of copies of $\mathbb{Z}$ consists of sequences in $\mathbb{Z}$ with a finite number of nonzero terms, while the countaby infinite Cartesian product of $\mathbb{Z}$ consists of all sequences in $\mathbb{Z}$.

3 Monopoles from Grand Unified Theories

Thanks to Kevin Zhou for this problem.

Grand unified theories (GUTs) generically predict the formation of magnetic monopoles in the early universe. The fact that we don't observe these monopoles today was one of the motivations for cosmic inflation. In this problem we show why GUTs make this prediction, using homotopy theory.

Point Defects

Homotopy groups can be used to detect topological defects in physical systems. Suppose the state of a 2D system is characterized by a field $\varphi(x) : \mathbb{R}^2 \to M$, where M is called the order parameter space. Now consider a loop in the plane, given by $\gamma : S^1 \to \mathbb{R}^2$. Then we can consider the field values we see as we traverse the loop, i.e. the function $f = \varphi \circ \gamma : S^1 \to M$.

(a) Prove that if the field is continuous everywhere, f is homotopic to the constant map.

Therefore, if f is *not* homotopic to the constant map, the field must contain a singularity, which we call a point defect. Moreover, we can classify point defects using the fundamental group $\pi_1(M)$. Point defects are "topological" features because they are not affected by continuous deformations of the field φ, which makes them more durable.

(b) Consider a planar magnet where the magnetization is a unit vector in the plane. Identify the order parameter space and show that there are point defects. How could a point defect be removed? Note: see Nakahara Section 4.9 if the terminology in this part is confusing.

Topological Defects

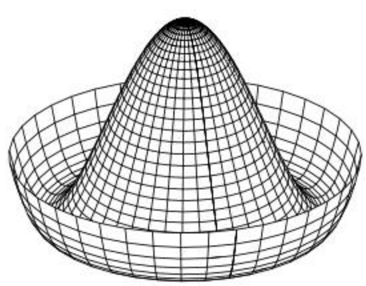
In a three-dimensional system, the fundamental group no longer detects point defects; it instead detects line defects, one-dimensional objects which pass through the loop. In superfluids, these are called vortices; in cosmology, they are called cosmic strings. In order to detect point defects, we instead need to enclose them with a sphere, so they are characterized by the second homotopy group $\pi_2(M)$.

(c) In three spatial dimensions, what kinds of defects do π_0 and π_3 detect?

(d) Consider a magnet in three dimensions, as in problem 1(b). What kinds of defects are present?

Symmetry Breaking

Spontaneous symmetry breaking is an important process that happened in the early universe. The classic example is a complex scalar field ϕ in a Mexican hat potential, shown below.



The theory has a rotational $U(1)$ symmetry since the potential doesn't depend phase, but the actual vacuum expectation value of the field is just a single complex number, which does not have $U(1)$ symmetry. We say the $U(1)$ symmetry has been broken.

In general, let the unbroken symmetry group be G, and the remaining symmetry group of the vacua be H. Then physically distinct vacua live in the order parameter space G/H, and we can extract topological information from it.

(e) Consider a set of four real scalar fields $\phi_1, \ldots, \phi_4$ where the potential only depends on

$$|\phi|^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2.$$

What is the unbroken symmetry group G?

(f) Suppose that the potential is a Mexican hat potential with minimum at $|\phi| = v$. What is the remaining symmetry group H?

(g) Is G/H a group?

Because symmetry breaking occurs in the early universe, topological defects can be left behind; they are detected by homotopy groups as described in the previous part of this question. The four kinds of defects and their corresponding homotopy groups are:

$$\pi_0(G/H) : \text{domain walls}, \quad \pi_1(G/H) : \text{cosmic strings},$$

$$\pi_2(G/H) : \text{monopoles}, \quad \pi_3(G/H) : \text{cosmic textures}.$$

In order to find the homotopy groups of G/H, we must use the **long exact sequence**

$$\ldots \to \pi_{n+1}(G/H) \to \pi_n(H) \to \pi_n(G) \to \pi_n(G/H) \to \pi_{n-1}(H) \to \ldots$$

The idea is similar to that of the chain complex we saw when dealing with homology. Each arrow represents a group homomorphism. Consider a small part of this sequence,

$$H \xrightarrow{f} G \xrightarrow{g} K.$$

We say the sequence is exact if

$$\text{im}(f) = \ker(g)$$

which implies that, like the chain complex, the composition of any two adjacent maps is the zero map. Moreover, the First Isomorphism Theorem (also introduced in chapter 3) tells that $K \cong G/H$.

(h) Using the long exact sequence and the table of homotopy groups in Nakahara, find the homotopy groups of G/H. What cosmological defects exist in this model?

Grand Unified Theories

A grand unified theory starts with a large gauge group G that is spontaneously broken to the gauge group of the Standard Model,

$$H = SU(3) \times SU(2) \times U(1).$$

The smallest gauge group that allows this is $G = SU(5)$, yielding the Georgi-Glashow model.

(i) Show that the Georgi-Glashow model predicts monopoles.

We do not have enough machinery to show these monopoles are *magnetic* monopoles, but we will in a few weeks.

4 Anyons and Braid Groups

Thanks to Saran Prembabu for this problem.

It is well known that in three dimensional space, particles can be classified as fermions or bosons. It is often said that fermionic wavefunctions ψ_f satisfy $\psi_f(x_1, x_2) = \langle x_1, x_2 | \psi_f \rangle = -\psi_f(x_2, x_1)$ while bosonic wavefunctions ψ_b satisfy $\psi_b(x_1, x_2) = \psi_b(x_2, x_1)$, where $|x_1, x_2\rangle$ is the localised state with one particle in x_1 and one particle in x_2. But this is a misleading abuse of notation, because really $|x_1, x_2\rangle = |x_2, x_1\rangle$: the operators that create particles at each point commute. This problem will describe the quantum statistics of bosons and fermions in terms of homotopy.

Configuration Spaces

First, a brief definition: a **configuration space** M_n^d is a topological space defined as $\frac{(\mathbb{R}^d)^n - \delta}{S_n}$, where $(\mathbb{R}^d)^n$ is the product of n copies of the topological space $\mathbb{R}^d$. $(\mathbb{R}^d)^n - \delta$ is the subset $\{(x_1, x_2, \ldots, x_n)\} \subset (\mathbb{R}^d)^n$ of ordered n-tuples of vectors of length d such that no two d-vectors in an n-tuple agree in every entry; i.e., all d-vectors in an n-tuple describe distinct points in $\mathbb{R}^d$. This is to say that no two particles coexist at the exact same point. Finally, the quotient by the symmetric group S_n means that $(x_1, x_2, \ldots, x_n)$ is identified with any permutation of the coordinates: $(x_\sigma(1), x_\sigma(2), \ldots, x_\sigma(n))$ for any $\sigma \in S_n$, since particles are identical.

(a) Draw M_2^1, M_1^2 and M_3^1.

(b) Describe how M_2^2 could look like on a 4-d Cartesian grid.

(c) All of the possible arrangements of n indistinguishable non-superimposable particles in d-dimensional space form a topological space. What is this topological space?

In reality, the argument of a wavefunction of multiple identical particles isnt an arbitrary ordered n-tuple, but rather a single point in a configuration space, for the reasons described above (no coexisting particles, particles being identical). Furthermore, the wavefunction is not actually a function, but a multi-valued quantity for each point in configuration space. This distinction matters when discussing fractional statistics.

Homotopy Groups of Configuration Spaces

Now we will study the topological properties of configuration spaces, proving a famous result by Professor Artin's dad. It is recommended that you think visually about these things.

(d) What is $\pi_1(M_n^1)$ for any n?

(e) What is $\pi_1(M_1^d)$ for any d?

(f) What is $\pi_1(M_2^3)$?

(g) What is $\pi_1(M_2^d)$ for $d > 3$? (an intuitive justification is sufficient here; the topological space here is hard to visualise, but as a hint try to use the result of part (c) and imagine what an S_1 loop in M_2^d means in this context)

(h) What is $\pi_1(M_n^d)$ for $n \geq 2$ and $d \geq 3$?

Braid Groups

Note that above we skipped the $d = 2$ case. That is because it is interesting. $\pi_2(M_n^2)$ is called the braid group B_n on n strands.

(i) Prove that $M_2^2 = \mathbb{R}^2 \times \mathcal{A}$, where $\mathcal{A}$ is a punctured disk.

(j) Find $\pi_1(M_2^2)$ using the product property of homotopy groups. Convince yourself that this makes sense when considering two particles going around each other in a plane. Note in particular that $\pi_1(M_2^2)$ does not fit the pattern suggested in part (g).

(k) Describe $\pi_1(M_3^2)$ in terms of some generators and relations. (Hint: $\alpha : [0,1] \to M_2^3$ is a function $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_n(t))$, where $\alpha_j : [0,1] \to \mathbb{R}^2$. Consider simultaneously graphing the functions $\alpha_1(t), \alpha_2(t), \alpha_n(t)$ on a 3-d coordinate system)

(l) Prove by induction that $\pi_1(M_n^2)$ (for $n \geq 2$) is an infinite group generated by $n-1$ generators $\sigma_1, \sigma_2, \ldots \sigma_{n-1}$, with the defining relations $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} = 1$ for any $1 \leq i \leq n-2$ and $\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} = 1$ for any $|i - j| \geq 2$. This is the presentation of the **braid group** on n strands.

Braid Groups and Quantum Statistics

Now we will learn why we randomly decided to compute various first homotopy groups in the above problems. Suppose a collection of identical particles initially has a wave function $\psi_a(q, 0)$ at time $t = 0$, where $q \in M_n^d$ is a configuration space point. Then at time $t = T$ the wave function is $\psi_a(q, T) = \int dq \langle q, T | q, 0 \rangle \psi_a(q, 0)$.

When $\pi_1(M_n^d)$ is trivial, $\langle q, T | q, 0 \rangle = \int \mathcal{D}q \exp(iS[q])$ (possibly multiplied by an arbitrary complex phase factor) where $\mathcal{D}q$ is an integration over all loops in M_n^d,. But when $\pi_1(M_n^d)$ is nontrivial, this *path integral* is actually a discrete sum of integrals from each homotopy equivalence class, and each equivalence class can have its own arbitrary phase factor. So, $\langle q, T | q, 0 \rangle = \sum_{\alpha \in \pi_1(M_n^d)} e^{i\rho(\alpha)} \int \mathcal{D}q \exp(iS[q])$ where $\rho : \pi_1(M_n^d) \to \mathbb{R}$. To be consistent with quantum mechanics, we can (but wont) prove that the only condition ρ must satisfy is that $\rho(\alpha \cdot \beta) = \rho(\alpha) + \rho(\beta)$ where $\cdot$ is the group operation in $\pi_1(M_n^d)$, i.e. ρ must form a one-dimensional additive representation of $\pi_1(M_n^d)$.

(m) Find all possible homomorphisms ρ if $d \geq 3$ and $n \geq 2$

(n) In the previous part, one of the homomorphisms corresponds to bosonic statistics and one of them corresponds to fermionic statistics. Which one is which?

(o) Find all possible homomorphisms ρ if $n = d = 2$

(p) Find all possible homomorphisms ρ if $d = 2$ and $n \geq 2$.

(q) Show that for identical particles in two dimensions starting with wave function ψ_a, it is possible to rearrange them according to the group action of one of the generators σ_j of the fundamental group so that you end up with the wave function $e^{i\nu\pi}\psi_a$, where ν can have any real value in $\mathbb{R}/2\mathbb{Z}$. The value of ν gives the "statistics" of these particles: $\nu = 0$ for bosons, $\nu = 1$ for fermions, and for fractional values the particles are called **anyons**.

5 Instantons and Excitons in 2D Condensed Matter Systems

Thanks to Mike Winston for this problem.

In condensed matter physics, there is something called the XY model. It covers a lot of 2D systems, like liquid crystals.

The basic idea is that the degrees of freedom of some system live in a 2D plane. We can for example consider a system of particles that each have some spin at every point, where the spins are required to be coplanar. Let's call this a map from $\mathbb{R}^2$ to S^1, to suggest using the machinery of homotopy groups (really homotopy sets, since we aren't going to endow this with a group operation).

(a) Prove that the homotopy set of the 2D spin system described above is trivial, in the sense that all maps $\mathbb{R}^2 \to S^1$ are homotopic to the constant map.

Though this set is trivial, we can consider an extension of the problem. Suppose that instead of $\mathbb{R}^2$ we have $\mathbb{R}^2$ minus a point; the **punctured plane**. In physics, this puncture would correspond to an excited point or **exciton** where the spin is not defined.

(b) What is the new homotopy set of this system?

(c) What is the homotopy set if there are multiple excitons at multiple spatial points?

Here is also a more difficult extension of the problem: take for granted that one can consider ground states of the weak force to correspond to maps from S^4 ($\mathbb{R}^4$ with the boundary identified) to $SU(2)$. $SU(2)$ has the same topology as S^3, and $\pi_4(S^3)$ is Z_2 to be proved below).

(d) Prove that $\pi_4(S^2)$ is Z_2.

(e) Argue that this means that an instanton (stable higher-energy perturbation of the ground state) for the weak interaction is its own antiparticle.