Nakahara Ch. 5 Problems

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Topics covered: Manifolds, calculus on manifolds, vector fields and one-forms, flows and Lie derivatives, one-parameter groups of transformations, differential forms and applications to Hamiltonian mechanics, integration on manifolds, Lie groups, left-invariant vector fields, Lie algebras, Lie group actions on manifolds, induced vector fields, adjoint representation of Lie groups.

As we begin introducing more of the geometry, applications to physics are more widespread. With the introduction of differential forms, we can now do calculus on arbitrary curved manifolds like those that appear in general relativity or in gauge theory. Differential forms will lead to the definition of de Rham cohomology, which ends up classifying topological invariants like magnetic monopole charge. Lie groups and Lie algebras are central to this discussion; typically either the manifold in question or the relevant group action in physics will be a Lie group because of the natural differentiable structure. A lot more of the exercises in Nakahara are now relevant, but I've tried to draw some good examples of my own. This chapter was a monster, so I have a lot of one-hit problems to cover the critical concepts.

1 "Short" "Math" Problems

(a) *Rewording of Nakahara Exercise 5.1.* At the beginning of Chapter 5, Nakahara introduces stereographic coordinates on the sphere projected from the North Pole. Show explicitly (by using the functional forms of the different stereographic projections and explicitly checking that the transition functions are C^∞) that the atlas defined by these two charts (coordinate systems) makes S^2 into a 2-dimensional smooth manifold.

Solution.
The stereographic projection from the sphere onto the plane from the North Pole is given by the coordinate functions:

$$x = \cot(\theta/2)\cos\phi, \qquad y = \cot(\theta/2)\sin\phi.$$

where θ is the angle from the z axis and ϕ is the angle around the z axis, as given on p.170 of Nakahara. To find the projection from the South pole, just take $\theta \to \pi - \theta$. This changes the cotangent to tangent, giving the new coordinates:

$$x' = \tan(\theta/2)\cos\phi, \qquad y' = \tan(\theta/2)\sin\phi.$$

The transition functions go from (x, y) to (x', y'). One can find them by inverting the above two. Using Mathematica, I find them to be:

$$x = \frac{x'}{(x')^2 + (y')^2}, \qquad y = \frac{y'}{(x')^2 + (y')^2}.$$

These are obviously C^∞ outside the origin in (x', y') coordinates. The origin in these coordinates is at $\theta = 0$. This is the North Pole, which is not in the intersection of the two coordinate systems, so we are safe: everywhere they are defined, the transition functions are smooth. □

(b) *Rewording of Nakahara Exercise 5.4.* Let $f : M \to N$ be a smooth map between manifolds. Show that for $\omega = \omega_\alpha dy^\alpha \in T^*_{f(p)}N$, the induced one-form $f^*\omega = \xi_\mu dx^\mu \in T^*_p M$ has components:

$$\xi_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}.$$

Note: this problem should be easy; you can essentially follow the discussion on p.186 with the placement of indices moved around. But it's intended to help you wade through the dense notation and terminology a bit.

Solution.
By definition, the pullback can be written in a coordinate basis:

$$(f^*\omega)_\mu V^\mu = \omega_\alpha (f_* V)^\alpha,$$

where $V \in T_p M$ and $\omega \in T^*_{f(p)}N$, c.f. equation (5.36) in Nakahara.

We are assuming the basis on N is given by the coordinates y^α and on M by coordinates x^μ above. But we know from (5.33) in Nakahara that the coordinates of $(f_* V)$ are:

$$(f_* V)^\alpha = V^\mu \frac{\partial y^\alpha}{\partial x^\mu}.$$

So the pullback is written:

$$(f^*\omega)_\mu V^\mu = \omega_\alpha V^\mu \frac{\partial y^\alpha}{\partial x^\mu},$$

that is, eliminating V^μ from both sides:

$$(f^*\omega)_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}.$$

Labeling $\xi_\mu = (f^*\omega)_\mu$ as in the problem statement, we have thus shown:

$$\xi_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu},$$

as claimed. □

(c) *Nakahara Exercise 5.7.* Let $M = \mathbb{R}^2$ and let $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Find the flow generated by X.

Solution.
We are looking for a parametric function $\sigma(x, y, t)$ which satisfies:

$$\frac{d\sigma}{dt} = X(\sigma).$$

That is, the flow is defined so that the "velocity" of the flow is the vector field X at all times. Note that a negative sign in front of the y would make this vector field generate rotations and the corresponding flow would be circular. By analogy we should expect the flow to be hyperbolic, given by:

$$\sigma(x, t) = (x \cosh t + y \sinh t, x \sinh t + y \cosh t).$$

We may verify:

$$\begin{aligned} \frac{d\sigma}{dt} &= (x \sinh t + y \cosh t, x \cosh t + y \sinh t) \\ X(\sigma) &= \left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \right) (x \cosh t + y \sinh t, x \sinh t + y \cosh t) \\ &= (y \cosh t + x \sinh t, y \sinh t + x \cosh t) \\ &= \frac{d\sigma}{dt} \end{aligned}$$

as claimed. □

(d) *Corrected Version of Nakahara Exercise 5.17.* Let $X, Y \in \mathcal{X}(M)$ and $\omega \in \Omega^r(M)$. Show that

$$i_{[X,Y]}\omega = X(i_Y\omega) - Y(i_X\omega) - d\omega(X, Y).$$

Note: there is an error in (5.83) in Nakahara; the last term is missing but certainly necessary (compare to 5.70 in Nakahara). Show also that i_X is an anti-derivation,

$$i_X(\omega \wedge \eta) = i_X\omega \wedge \eta + (-1)^r \omega \wedge i_X\eta$$

and nilpotent

$$i_X^2 = 0$$

Use the nilpotency to prove

$$[\mathcal{L}_X, i_X]\omega = 0.$$

Note: a *derivation* is basically an operator that obeys an analogue of the product rule. An *anti-derivation* is one that (possibly) picks up a sign in the product rule.

Solution.
First, recall that the commutator can be written in coordinates:

$$[X, Y]^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu.$$

Also a notational clarification: when we write a vector field as acting on an object f, we are referring to expanding out X in the basis for the tangent space i.e.:

$$X(f) = X^\mu \partial_\mu f.$$

This holds even when f is not a 0-form.

The interior product, by the definition (5.79), is thus (leaving the basis implicit):

$$\begin{aligned}
i_{[X,Y]}\omega &= \frac{1}{(r-1)!}(X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu)\omega_{\mu\mu_2\ldots\mu_r} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r} \\
&= X^\nu \partial_\nu \left(\frac{1}{(r-1)!} Y^\mu \omega_{\mu\mu_2\ldots\mu_r} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r} \right) - Y^\nu \partial_\nu \left(\frac{1}{(r-1)!} X^\mu \omega_{\mu\mu_2\ldots\mu_r} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r} \right) \\
&- \frac{1}{(r-1)!}(X^\nu Y^\mu - X^\mu Y^\nu)\partial_\nu \omega_{\mu\mu_2\ldots\mu_r} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r} \\
&= X(i_Y\omega) - Y(i_X\omega) - d\omega(X,Y)
\end{aligned}$$

We have dropped the terms above involving the derivative operating on the $r-1$-form basis; since the resulting $\delta_\nu^{\mu_i}$ terms are symmetric in their indices but the coefficients are antisymmetric, these will fall out. The first result is thus established.

To show the anti-derivation property (letting $\eta \in \Omega^s(M)$), recall first that (suppressing the basis for purposes of space; it should be clear):

$$(\omega \wedge \eta)_{\mu_1\ldots\mu_r\nu_1\ldots\nu_s} = \sum_{P \in S_{r+S}} \mathrm{sgn}(P)\omega_{P(\mu_1)\ldots P(\mu_r)}\eta_{P(\nu_1)\ldots P(\nu_s)}$$

Taking the interior product, we find:

$$\begin{aligned}
i_X(\omega \wedge \eta) &= \frac{1}{(r+s-1)!} X^\mu (\omega \wedge \eta)_{\mu\mu_2\ldots\mu_r\nu_1\ldots\nu_s} \\
&= \frac{1}{(r+s-1)!} X^\mu \sum_{P \in S_{r+s}} \mathrm{sgn}(P)\omega_{P(\mu)\ldots P(\mu_r)}\eta_{P(\nu_1)\ldots P(\nu_s)}
\end{aligned}$$

The right-hand side can be divided into terms of two types: those where $P(\mu_i) = \mu$ and those where $P(\nu_i) = \mu$, i.e. those where the μ index appears on ω or on η. If it appears on η, the sign of the permutation gains a factor of $(-1)^r$ from the r transpositions it takes to move μ through all of the μ_i.

If the μ ends up on an ω, there are $(r+s-1)!$ ways of permuting the remaining indices, but the $(r-1)!$ ways of permuting within ω do not give unique terms. The same is true of η, where the $(s-1)!$ ways of permuting indices within η are not unique. The sum thus separates:

$$\begin{aligned}
i_X(\omega \wedge \eta) &= \frac{1}{(r+s-1)!} X^\mu \Bigg(\sum_{P \in S_{r+s-1}} \frac{(r+s-1)!}{s!(r-1)!} \omega_{\mu P(\mu_2)\ldots P(\mu_r)}\eta_{P(\nu_1)\ldots P(\nu_s)} \\
&+ (-1)^r \frac{(r+s-1)!}{r!(s-1)!} \omega_{P(\nu_1)P(\mu_2)\ldots P(\mu_r)}\eta_{\mu P(\nu_2)\ldots P(\nu_s)} \Bigg) \\
&= i_X\omega \wedge \eta + (-1)^r \omega \wedge i_X\eta
\end{aligned}$$

which establishes the anti-derivation fact.

To prove nilpotency, we have:

$$i_X^2\omega = \frac{1}{(r-1)!(r-2)!} X^\mu X^\nu \omega_{\nu\mu\mu_3\ldots\mu_r} dx^{\mu_3} \wedge \ldots \wedge dx^{\mu_r} = 0,$$

where equality to zero follows from the antisymmetry of the basis (since $X^\mu X^\nu$ is symmetric).

Proving the last part is fortunately easy. We use the result that $\mathcal{L}_X\omega = (di_X + i_X d)\omega$. Using the nilpotency of i_X:

$$\mathcal{L}_X i_X \omega = (di_X + i_X d) i_X \omega = i_X d i_X \omega = i_X(\mathcal{L}_X - i_X d)\omega = i_X \mathcal{L}\omega,$$

as claimed. □

(e) Consider a vector in the $2n$-dimensional phase space of a dynamical system: $z = (q_1, \ldots, q_n, p_1, \ldots, p_n)^T$. If the dynamical system is quantum-mechanical, the canonical commutation relation can be written in terms of the symplectic two-form ω (expressed as a $2n \times 2n$ matrix Ω by writing in the tensor product basis) via:

$$[z_i, z_j] = i\hbar\Omega_{ij}.$$

Suppose the Hamiltonian is quadratic in the phase-space coordinates, i.e.:

$$H = \frac{1}{2} z^T K z = \frac{1}{2} z_i K_{ij} z_j$$

for K a symmetric bilinear form on the coordinates.

Note that Ω can be written in the block matrix form:

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Show that the Heisenberg equation $i\hbar \frac{d}{dt} z_i = [z_i, H]$ becomes:

$$\frac{dz}{dt} = \Omega K z,$$

and show that the time-evolution of this system is equivalent to an action of the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ on the phase space. Note that $\mathrm{Sp}(2n, \mathbb{R})$ is the set of $2n \times 2n$ real matrices A that preserve Ω under conjugation: $A^T \Omega A = \Omega$.

Note that the commutators here refer to the commutation as operators on the Hilbert space of the QM dynamical system, not commutation as finite matrices or similar.

Solution.
The Heisenberg equation for the quadratic Hamiltonian can be written (using the symmetry of K):

$$\begin{aligned}
\frac{dz_i}{dt} &= \frac{1}{i\hbar}[z_i, \frac{1}{2} z_j K_{jk} z_k] = \frac{1}{2i\hbar} K_{jk}([z_i, z_j] z_k + z_j [z_i, z_k]) \\
&= \frac{1}{2} K_{jk}(i\hbar \Omega_{ij} z_k + i\hbar z_j \Omega_{ik}) \\
&= \frac{1}{2}(\Omega_{ij} K_{jk} z_k + \Omega_{ik} K_{kj} z_j) \\
&= (\Omega K z)_i
\end{aligned}$$

So the new form of the Heisenberg equation is thus verified. Note, by the way that this is equivalent to Hamilton's equations of motion: If we consider i in $1 \dots n$, have:

$$q_i = \Omega_{ij} K_{jk} z_k = \Omega_{ij} \frac{\partial}{\partial z_j} (\frac{1}{2} z_j K_{jk} z_k) = \frac{\partial H}{\partial p_i}$$

Similarly if i is in $n+1 \dots 2n$, the same computation holds with a sign flip from Ω, giving the usual:

$$p_i = -\frac{\partial H}{\partial q_i}$$

Showing that the time-evolution is equivalent to an action of the real symplectic group on the phase space is saying exactly that we want the action of $\frac{d}{dt}$ to be given by an element of the Lie algebra A. So $A = \Omega K$ must be in the Lie algebra of the symplectic group, if this is true. As a consequence, we would have $z(t) = e^{tA} z(0)$ i.e. time evolution is given by the action of a one-parameter subgroup (a flow, specifically the *Hamiltonian flow*) of the symplectic group.

We can employ the usual trick to find the condition for a matrix to be an element of the Lie algebra for the symplectic group. Let $I + tA$ be an element of the symplectic group, A in the Lie algebra (literally this is what it means for the Lie algebra to be the tangent space at the identity). Then $I + tA$ satisfies:

$$(I + tA)^T \Omega (I + tA) = 0 \implies A^T \Omega + \Omega A = 0.$$

We thus verify, using the symmetry of K and the facts that $\Omega^2 = -I$, $\Omega^T = -\Omega$:

$$(\Omega K)^T \Omega + \Omega(\Omega K) = K^T \Omega^T \Omega + \Omega^2 K = -K(\Omega)^2 + \Omega^2 K = K - K = 0$$

Therefore, the time-evolution is indeed equivalent to an action of the real symplectic group on the phase space.

□

(f) *Rewording of Nakahara Exercise 5.20.* Show that the group $SO(1,3)$, i.e., the Lorentz group, is non-compact and has four connected components according to the sign of the determinant and the sign of the $(0,0)$ entry.

Note: noncompact groups cannot have nontrivial irreducible unitary representations. This is therefore an interesting and relevant fact in QFT, where the "spin" of a particle refers to the (finite-dimensional) representation of the Lorentz group under which it transforms.

Solution.
The Lorentz group can be embedded in $\mathbb{R}^{16}$ viewing it as a set of four by four matrices. Then since the velocities of boosts are not closed above ($v < c$, strictly), the Heine-Borel theorem tells us that the Lorentz group will not be compact, since we do not have a closed, bounded subset.

The four connected components can be reached by the discrete Lorentz transformations of parity P and time reversal T. One can easily show that these preserve the Minkowski metric η and therefore are indeed Lorentz transformations. The four components are therefore the component connected to the identity, the identity component times P, the identity component times T, and the identity component times PT. The T multiplication flips the sign of the $(0, 0)$ entry, and the parity transformation flips the sign of the three spatial eigenvalues, thus changing the sign of the determinant. □

(g) *Rewording of Nakahara Exercise 5.22.* Let

$$c(s) = \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be a curve in $SO(3)$. Find the tangent vector to this curve at I_3 (the identity).

Solution.
The tangent vector to the curve at I_3 is just the derivative evaluated at $s = 0$, noting that $s = 0$ corresponds to the identity on the curve:

$$\left.\frac{dc}{ds}\right|_{s=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If we write:

$$c(s) = I_3 + s\left.\frac{dc}{ds}\right|_{s=0} + \mathcal{O}(s^2),$$

then $c(s)$ to first order in s is just the infinitesimal transformation given by the curve c found by Taylor expanding each of the entries.

□

(h) *Rewording of Nakahara Exercise 5.24.* Verify by explicit calculation that

$$A = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

represents a rotation about the z-axis by θ and

$$B = \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix}$$

represents a boost along the z-axis with velocity $v = \tanh \alpha$.

Solution.
We act with the given matrices on M_4, Minkowski space written in the Pauli basis $x^\mu \sigma_\mu$. Then we have:

$$\begin{aligned} AXA^\dagger &= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} (x^0 + x^3)e^{i\theta/2} & (x^1 - ix^2)e^{-i\theta/2} \\ (x^1 + ix^2)e^{i\theta/2} & (x^0 - x^3)e^{-i\theta/2} \end{pmatrix} \\ &= \begin{pmatrix} x^0 + x^3 & (x^1 - ix^2)e^{-i\theta} \\ (x^1 + ix^2)e^{i\theta} & x^0 - x^3 \end{pmatrix} \end{aligned}$$

Written as a transformation of coordinates, x^0 and x^3 are unchanged. x^1 and x^2 have transformed so that:

$$\begin{aligned} x^1 &\to \frac{1}{2}(x^1 - ix^2)e^{-i\theta} + \frac{1}{2}(x^1 + ix^2)e^{i\theta} = x^1 \cos\theta - x^2 \sin\theta \\ x^2 &\to -\frac{1}{2i}(x^1 - ix^2)e^{-i\theta} + \frac{1}{2i}(x^1 + ix^2)e^{i\theta} = x^1 \sin\theta + x^2 \cos\theta \end{aligned}$$

This is clearly a rotation about the z-axis by θ. The analogous computation for B gives:

$$BXB^\dagger = \begin{pmatrix} (x^0 + x^3)e^{\alpha} & x^1 - ix^2 \\ x^1 + ix^2 & (x^0 - x^3)e^{-\alpha} \end{pmatrix}$$

As a transformation of coordinates, we therefore have:

$$\begin{aligned} x^0 &\to \frac{1}{2}(x^0 + x^3)e^{\alpha} + \frac{1}{2}(x^0 - x^3)e^{-\alpha} = x^0 \cosh\alpha + x^3 \sinh\alpha \\ x^3 &\to \frac{1}{2}(x^0 + x^3)e^{\alpha} - \frac{1}{2}(x^0 - x^3)e^{-\alpha} = x^0 \sinh\alpha + x^3 \cosh\alpha \end{aligned}$$

which is a Lorentz boost on the z-axis with velocity $v = \tanh\alpha$ (if this is not obvious to you from previous experience, try writing out γ and $\beta\gamma$ explicitly). □

(i) *Rewording of Nakahara Exercise 5.27.* The Lie group $SO(2)$ acts on $M = \mathbb{R}^2$ by rotation. Let

$$V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

be an element of $\mathfrak{so}(2)$. Show that

$$\exp(tV) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and find the induced flow through $x = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Show that $V^\sharp|_x = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

Solution.
We first compute the exponential. Note that $V^2 = -I$, so using the Taylor expansions of sine and cosine, we find:

$$\begin{aligned}
\exp(tV) &= \sum_{n=0}^{\infty} \frac{(tV)^n}{n!} = \sum_{n \text{ odd}} \frac{(tV)^n}{n!} + \sum_{m \text{ even}} \frac{(tV)^m}{m!} \\
&= \sum_{j=0}^{\infty} \frac{t^{2j+1}V^{2j}V}{(2j+1)!} + \sum_{k=0}^{\infty} \frac{t^{2k}V^{2k}}{(2k)!} \\
&= V\sum_{j=0}^{\infty} \frac{t^{2j+1}(-1)^j}{(2j+1)!} + I\sum_{k=0}^{\infty} \frac{t^{2k}(-1)^k}{(2k)!} \\
&= V\sin t + I\cos t \\
&= \begin{pmatrix} 0 & -\sin t \\ \sin t & 0 \end{pmatrix} + \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} \\
&= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}
\end{aligned}$$

as claimed. The induced flow is clearly just the rotation:

$$\sigma(t, \vec{x}) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos t - y\sin t \\ x\sin t + y\cos t \end{pmatrix}$$

The induced vector field, from the definition (5.160) in Nakahara, is:

$$V^\sharp|_x = \frac{d}{dt}\exp(tV)x\bigg|_{t=0} = \begin{pmatrix} -x\sin t - y\cos t \\ x\cos t - y\sin t \end{pmatrix}\bigg|_{t=0} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

which written in the usual basis for the tangent space is:

$$V^\sharp|_x = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

as stated. □

(j) *Rewording of Nakahara Exercise 5.28.* Show that $\mathrm{Ad}_a : g \mapsto aga^{-1}$ for $a, g \in G$ is a homomorphism. Define a map $\sigma : G \times G \to G$ by $\sigma(a, g) \equiv \mathrm{Ad}_a g$. Show that $\sigma(a, g)$ is an action of G on itself.

Note: I have picked the more common convention for capitalization here where capital Ad acts on the group and the lowercase ad acts on the algebra. But note that this is opposite Nakahara's convention!

Solution.
First, we show that this is a homomorphism:

$$\mathrm{Ad}_a(gh) = agha^{-1} = aga^{-1}aha^{-1} = \mathrm{Ad}_a(g)\mathrm{Ad}_a(h).$$

Well, at least that part was easy. Now we show that $\sigma(a, g)$ is an action of G on itself. We just need σ to satisfy the conditions (5.141) in Nakahara. So let's check:

$$\sigma(e, g) = \mathrm{Ad}_e g = ege^{-1} = g,$$
$$\sigma(a, \sigma(b, g)) = a\sigma(b, g)a^{-1} = abgb^{-1}a^{-1} = (ab)g(ab)^{-1} = \sigma(ab, g).$$

So indeed this does define an action of G on itself. There are interesting connections here between the adjoint map in terms of group elements and the Lie bracket, which let you go between representations of Lie groups and of Lie algebras. Unfortunately I don't have the time to explore them further here. □

2 Lie Group Actions on Spheres

Minor modification of Problems 2 and 3 from Pset 4, 18.755 Fall 2014.

(a) Consider the three vector fields on the circle

$$H = \frac{d}{d\theta}, \quad U = \cos(\theta)\frac{d}{d\theta}, \quad V = \sin(\theta)\frac{d}{d\theta}$$

(b) Prove that the Lie brackets of these vector fields are

$$[H, U] = -V, \quad [H, V] = U, \quad [U, V] = H.$$

(c) Find elements H_1, U_1, and V_1 in $\mathfrak{sl}(2, \mathbb{R})$ satisfying these same bracket relations.

(d) Is there an action of $SL(2, \mathbb{R})$ on the circle corresponding to the previous three parts?

(e) For any integer $n \geq 1$, consider the three vector fields on the circle

$$H_n = \frac{d}{d\theta}, \quad U_n = \cos(n\theta)\frac{d}{d\theta}, \quad V_n = \sin(n\theta)\frac{d}{d\theta}.$$

Prove that the Lie brackets of these vectors fields are

$$[H_n, U_n] = -nV_n, \quad [H_n, V_n] = nU_n, \quad [U_n, V_n] = nH_n.$$

(f) Find elements H_n, U_n, and V_n in $\mathfrak{sl}(2, \mathbb{R})$ satisfying these same bracket relations.

(g) Is there an action of $SL(2, \mathbb{R})$ on the circle explaining the preceding two parts?

(h) Consider the three vector fields on the two-sphere

$$H' = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad U' = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad V' = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

(i) Prove that the Lie brackets of these vector fields are

$$[H', U'] = V', \quad [H', V'] = -U', \quad [U', V'] = H'.$$

(j) Find elements H_1', U_1', and V_1' in $\mathfrak{su}(2)$ (complex trace zero skew-Hermitian matrices) satisfying these same bracket relations.

(k) Is there an action of $SU(2)$ on the real two-sphere explaining the preceding two problems?

(l) Is there a way of nontrivially extending this action to include an n as in the case of the $SL(2,\mathbb{R})$ action on the circle? If so, how; if not, why not?

3 Differential Forms and Electromagnetism

Extension of Nakahara Problem 5.2

Let M be the Minkowski four-spacetime. Define the action of a linear operator $* : \Omega^r(M) \to \Omega^{4-r}(M)$ by:

$$\begin{aligned}
r=0: &\quad *1 = -dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3; \\
r=1: &\quad *dx^i = -dx^j \wedge dx^k \wedge dx^0 \qquad *dx^0 = -dx^1 \wedge dx^2 \wedge dx^3; \\
r=2: &\quad *dx^i \wedge dx^j = dx^k \wedge dx^0 \qquad *dx^i \wedge dx^0 = -dx^j \wedge dx^k; \\
r=3: &\quad *dx^1 \wedge dx^2 \wedge dx^3 = -dx^0 \qquad *dx^i \wedge dx^j \wedge dx^0 = -dx^k; \\
r=4: &\quad *dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = 1;
\end{aligned}$$

where (i,j,k) is an even permutation of $(1,2,3)$. The vector potential A. and the electromagnetic tensor F are defined as in example 5.11 (page 200 in Nakahara), i.e. $A = (\phi, \vec{A}) = A_\mu dx^\mu$ and $F = dA$. $J = J_\mu dx^\mu = \rho dx^0 + j_k dx^k$ is the current one-form.

Note: this is effectively (up to a constant) the *Hodge star operator*, more commonly called *Hodge dual*, mentioned in our last meeting, which gives a generalization of the cross product.

(a) Show that the Bianchi identity $dF = 0$ (which follows from the nilpotency of the exterior derivative, since $dF = d^2A = 0$) reduces to $\nabla \cdot B = 0$ and $\frac{\partial B}{\partial t} = -\nabla \times E$. Note that if $F = dA$ does not hold true everywhere, then we may have magnetic monopoles, as we saw several problem sets ago (since then F is not everywhere exact, i.e. dF is not everywhere zero).

(b) Write down the equation $d*F = *J$ and verify that it reduces to two of the Maxwell equations $\nabla \cdot E = \rho$ and $\nabla \times B - \frac{\partial E}{\partial t} = j$.

(c) Show that the identity $0 = d(d*F) = d*J$ reduces to the charge conservation equation

$$\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot j = 0.$$

(d) Show that the Lorentz condition $\partial_\mu A^\mu$ is expressed as $d*A = 0$.

OK, time to make the problem a bit more interesting. A *norm* on a vector space V in analysis is a function $V \to \mathbb{R}$ which obeys the following properties (let parallel bars denote the norm): 1. $|av| = a|v|$ for $a \in \mathbb{C}$ and $v \in V$. 2. $|v+w| \le |v| + |w|$ for $v, w \in V$. 3. $|v| = 0$ implies $v = 0$. Note: the below problems are particularly challenging. It will help to write things out in components.

(e) Show that the function defined by

$$f(v) = |v| = \int_M v \wedge *v$$

is a norm on the set of 2-forms on M.

(f) Show that the variation (as an action) of

$$|F| = \int_M F \wedge *F$$

yields the two vacuum Maxwell equations.

(g) On the other hand, show that the other possible kinetic term for the EM field, $F \wedge F$ is a total derivative. What term is it the total derivative of? *(This problem is a modification of a problem part from 8.324).*

Note: the answer in (g) is the Abelian version of what is called a *Chern-Simons 3-form*, and is related to a set of topological invariants we will get to much later called the *Chern characters* of the one-form A. It defines the action of *Chern-Simons theory*, for which the dynamics are trivial: $F = 0$, but gauge-inequivalent potentials A exist that lead to the same F (in the non-Abelian generalization) which ends up being a consequence of the fact that A lives in a manifold with nontrivial topology. Chern-Simons theory is physically useful in condensed matter theory in the fractional quantum Hall effect, and appears elsewhere in string theory and QFT. More on this later...

4 Lie Algebras and the Lorentz Group

A friendly introduction to your neighborhood fermion. Modification and extension of Peskin and Schroeder Problem 3.1.

The *Lorentz algebra* $\mathfrak{so}(1,3)$ is generated by the operators:

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu).$$

We assume the relevant metric is 4D Minkowski with $(+,-,-,-)$ signature. The algebra generated by these differential operators is specified by the commutation relations:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}).$$

Any matrix representation of the Lorentz algebra must satisfy these relations.

(a) Define the generators of rotations and boosts as:

$$L^i = \frac{1}{2}\epsilon^{ijk} J^{jk}, \qquad K^i = J^{0i}.$$

Write the commutation relations of these (vector) operators explicitly.

Thanks to Semon Rezchikov for some clarification on the following discussion. Now we introduce a "complexification" of the Lorentz algebra $\mathfrak{so}(1,3)$ by allowing complex combinations of elements. The complexification of $\mathfrak{so}(1,3)$ is isomorphic to $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

(b) We will here demonstrate the above fact explicitly. Define the complex combinations:

$$J_+ = \frac{1}{2}(L + iK), \qquad J_- = \frac{1}{2}(L - iK).$$

Show that J_+ and J_- generate two commuting copies of the $\mathfrak{su}(2)$ algebra (the commutation relations of angular momentum).

(c) The finite-dimensional representations of the rotation groups correspond precisely to the allowed values of angular momentum, integers or half-integers. The previous result thus implies that all finite-dimensional representations of the Lorentz group are indexed by pairs of integers/half-integers corresponding to the two copies of $\mathfrak{su}(2)$. Using the fact that $J = \frac{\sigma}{2}$ in the spin-(1/2) representation of angular momentum, where the σ^i are the Pauli matrices, write explicitly the transformation laws of the 2-component objects transforming according to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group.

Note: fermions, i.e. electrons, are so-called *Dirac bispinors* which transform in a *reducible* $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation comprised of the above two representations.

A point x in 4D Minkowski space M_4 may be represented as on p.217 of Nakahara by the 2×2 Hermitian matrix $X = x^\mu \sigma_\mu$ where $\sigma_\mu = (I, \vec{\sigma})$. Note that the determinant of this matrix is exactly the Minkowski norm.

Now take an arbitrary matrix A in $SL(2, \mathbb{C})$, and define an action on M_4 by:

$$\sigma(A, X) = AXA^\dagger.$$

(d) Show that this action preserves the Minkowski norm.

(e) Show that $SL(2, \mathbb{C})$ is a double cover of the Lorentz group $SO(1, 3)$.

(f) Show that the transformation laws in part (c) of this question are elements of $SL(2, \mathbb{C})$. Let A correspond to the matrix transformation law of the $(0, \frac{1}{2})$ representation. Show explicitly that x^μ transforms as a 4-vector under the given action on M_4.

(g) Show that $SL(2, \mathbb{C})$ is simply connected and that therefore it is the universal covering group of $SO(1, 3)$.

(h) The previous parts have shown that $SL(2, \mathbb{C})/\mathbb{Z}_2 = SO(1, 3)$ and that $SL(2, \mathbb{C})$ is the universal covering group of $SO(1, 3)$. Consequently, what is the fundamental group of $SO(1, 3)$?