Nakahara Ch. 6 Problems

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Topics covered: Stokes' theorem, de Rham cohomology, de Rham's thoerem, Poincaré's lemma, Poincaré duality, Cohomology rings and the Künneth formula, Pullback of de Rham cohomology groups, homotopy and the first cohomology group.

Using cohomology, the main algebraic and topological results from homology and homotopy can be connected with analysis and Lie theory. Differential forms on manifolds turn out to be dual to chains on manifolds via de Rham's theorem; the duality is provided by integration. In high energy physics, differential forms and their resulting cohomology are ubiquitous as different types of potentials and corresponding fields, from electromagnetism up through Yang-Mills theory and string theory.

Thanks to Semon Rezchikov for clarifying why Nakahara switches to $\mathbb{R}$-coefficients to discuss singular homology and cohomology despite using $\mathbb{Z}$-coefficients previously in the rest of the book. The reason is that the isomorphism of homology and cohomology groups does not hold up with $\mathbb{Z}$-coefficients; each gets a different factor of torsion subgroups, and even orientable manifolds that could be found in the wild can have nontrivial torsion subgroups.

1 Short Math Problems

(a) *Rewording of Nakahara Exercise 6.5.* Let $M = M_1 \times M_2$ be a product manifold. Show that $\chi(M) = \chi(M_1) \cdot \chi(M_2)$. Hint: use the Künneth formula.

(b) *Rewording of Nakahara Example 6.7* Define $\Omega = \sin\theta d\theta \wedge d\phi$ as a two-form on S^2. Verify that Ω is closed, and prove using Stokes' theorem that Ω is not exact. However, show also that Ω can formally be written as the exterior derivative of a one-form. Explain why Ω is still not exact despite this formal expression.

(c) Let A be the vector field on $\mathbb{C}^n$ defined by $A(\vec{z}) = i\vec{z}$, $z^k = x^k + iy^k$. Find a function $f : \mathbb{C}^n \to \mathbb{R}$ such that $\omega(A, V) = df(V)$ for all vector fields V, where $\omega = \sum_{k=1}^{n} dx^k \wedge dy^k$ is the standard symplectic form.

(d) Compute the de Rham cohomology groups of $SU(2)$ and $SU(3)$ (directly, not by appealing to isomorphism with homology groups).

Parts (c) and (d) taken from problems from MATH 4B03, McMaster University, http://ms.mcmaster.ca/minoo/Math4B03.html, Problem Sets 4 and 5.

2 Cohomology and the Magnetic Monopole

The topological setting of the magnetic monopole is the space $\mathbb{R}^3 \backslash \{0\}$, since the point containing the monopole is singular (in terms of the magnetic field or vector potential) and must be removed from the space.

(a) Prove that, in the absence of a magnetic monopole, a single vector potential suffices to describe the magnetic field everywhere.

Solution.
In the absence of the magnetic monopole, the topology is $\mathbb{R}^3$ since no point is removed. Since the first cohomology group of $\mathbb{R}^3$ is trivial, all one-forms are equivalent to the zero form and all differ by a globally defined exact form. So all vector potential one-forms are globally gauge-equivalent, so a second vector potential would be redundant. □

(b) We have shown in previous problem sets that in the presence of a single magnetic monopole, two vector potentials suffice to describe the field everywhere, although a single cannot. Prove that these two vector potentials *must* be locally gauge-equivalent, i.e. differ by the gradient of a scalar function (i.e. a (locally) exact one-form).

Solution.
In this part and the next part, we use the fact that the sphere S^2 is a deformation retract of $\mathbb{R}^3 \backslash \{0\}$. Deformation retraction preserves cohomology, so we can feel free to work with S^2 for the purposes of cohomology.

Now, the first cohomology group of S^2 is trivial. So all one-forms on S^2 must differ by an exact form. So we're done, right? It seems like two vector-potential one-forms must differ by an exact one-form.

Unfortunately, this is not correct. The reason is because the vector potentials are not globally defined for the magnetic monopole. One of them misses the North pole and one of them misses the South pole. The region on which both are defined is S^2 missing these two points. But this space deformation retracts to the circle S^1, which has first de Rham cohomology group of $\mathbb{R}$. So in fact the two vector potentials cannot be globally gauge equivalent! Of course, they will be *locally* gauge equivalent - this is a consequence of a corollary of the Poincaré lemma; since both vector potential one-forms define the same field strength, their difference is closed and thus locally exact.

Note that the argument above is not an all-encompassing proof. To make it fully rigorous we would have to deal with a lot of annoying details, I think. But one can note that no vector potential can be defined over the entirety of S^2 here, since this contradicts Stokes' theorem using the fact that S^2 is without boundary. The real work would have to be in dealing with vector potentials that are defined on even less than S^2 minus a point! □

We previously said that the magnetic field due to the monopole is $\vec{B} = \frac{g}{r^2}\hat{r}$. Let F be the field-strength two-form associated with this magnetic field (the Faraday tensor is a matrix; you can write it in a tensor product basis and convert this to a wedge product basis i.e. two-form basis, see p. 196).

(c) Show that $g = \frac{1}{4\pi}\int_{S^2} F$. Why can the integration be defined over S^2 instead of $\mathbb{R}^3\backslash\{0\}$?

Solution.
We have discussed in the previous part why the integration can be defined over S^2, due to the deformation retraction – any integral-defined topological invariant will be preserved (since this is given by the cohomology, loosely). There is a leap from saying cohomology is preserved to saying that integrals of cohomology classes are preserved; one can make this leap by proving theorems about the Chern classes later discussed, which *are* directly integral-defined quantities.

The Faraday tensor here (as a two-form) would be (in spherical coordinates; somewhat tedious to derive):

$$F = r^2 \sin\theta B_r (d\theta \otimes d\phi - d\phi \otimes d\theta) = r^2 \sin\theta B_r d\theta \wedge d\phi$$

Integrating, we have:

$$\frac{1}{4\pi}\int_{S^2} F = \frac{1}{4\pi}\int_{S^2} r^2 \sin\theta \frac{g}{r^2} d\theta \wedge d\phi = \frac{g}{4\pi}\int \sin\theta d\theta d\phi = g$$

□

(d) Relatedly, what de Rham cohomology group does g index? Later, we will see that g is also equal to a different topological invariant called the *first Chern class* of the $S^2 \times U(1)$ *fiber bundle*. The approach is convenient in that it allows us to construct similar topologically protected charges knowing only the gauge group of a theory (recall that $U(1)$ is the gauge group of electromagnetism).

Solution.
The integer g is given by the integral of the field-strength two-form over S^2 which is closed and without boundary. Since it is without boundary, the integral of an exact two-form over S^2 is trivial.

The closed two-form F is defined everywhere over S^2. Shifting it by an exact two-form does not change the integral that defines g by the previous paragraph. So g indexes $H^2(S^2) = H^2(\mathbb{R}^3\backslash\{0\})$. □

3 Trivializing the Cohomology

Rewording of Problem 1 from http://bohr.physics.berkeley.edu/classes/222/hws/hw8.pdf. I believe the problem is actually much easier than they intended, so I have removed the hint – let me know if this is not the case.

Let M be a manifold of dimension D and suppose there is a deformation retract of M onto a submanifold of dimension $d < D$. Show that $H^r(M) = \{0\}$ for $r > d$.