Explorations in Mathematical Inquiry

Penn Summer Prep Program

Matthew DeCross

July 25, 2017

Overview

- Introduction
- Schedule/Format
- General Communication Techniques
- LaTeX Introduction
- Presentation Techniques
- Beamer Introduction
- Theorems and Proofs
- Group Presentation and Topics
- Conclusion/Questions

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We're Here to Learn How Math is Done

- Goal: understand the math research and communication process through experience

Wait, How *is* Math Done?

- Learn about problems
- Answer problems
- Present work (written/oral)

Find Math Papers at the arXiv and Journals

Cornell University Library

We gratefully acknowledge support from
the Simons Foundation
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arXiv.org

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Search or Article ID | All papers

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Open access to 1,279,823 e-prints in Physics, Mathematics, Computer Science, Quantitative Biology, Quantitative Finance and Statistics

Subject search and browse: Physics | Search | Form Interface | Catchup

20 Apr 2017: Applied Physics subject area added to arXiv
10 Mar 2017: New members join arXiv Member Advisory Board
06 Mar 2017: arXiv Scientific Director Search
10 Feb 2017: Attention Submitters: our TeX processing system has been updated
See cumulative "What's New" pages. Read robots beware before attempting any automated download

Mathematics

- Mathematics (**math** new, recent, find)
  includes (see detailed description): Algebraic Geometry; Algebraic Topology; Analysis of PDEs; Category Theory; Classical Analysis and ODEs; Combinatorics; Commutative Algebra; Complex Variables; Differential Geometry; Dynamical Systems; Functional Analysis; General Mathematics; General Topology; Geometric Topology; Group Theory; History and Overview; Information Theory; K-Theory and Homology; Logic; Mathematical Physics; Metric Geometry; Number Theory; Numerical Analysis; Operator Algebras; Optimization and Control; Probability; Quantum Algebra; Representation Theory; Rings and Algebras; Spectral Theory; Statistics Theory; Symplectic Geometry

Several Steps of Doing Math

- Find good questions to ask
- Figure out ways to investigate the questions
- Convince yourself and others of what you learn

Tell People About What You Learn

- Typeset mathematical work in LaTeX and Beamer
- Work on techniques for improving technical writing (journals)
- Practice with technical oral presentation (group meetings, seminars, colloquia)

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Overview of the Schedule

- Tuesday: Math communication bootcamp
- Wed - Next Tues: Sandbox problems
- Next Wed - Thurs: Research topic, develop presentation
- Next Fri: Group presentations

Module Assignment Schedule

- Small (15-20 min) assignments due daily
- Short (2 page) paper on sandbox problem due next Tues (more on this later)
- Group presentation next Friday (more on this later)

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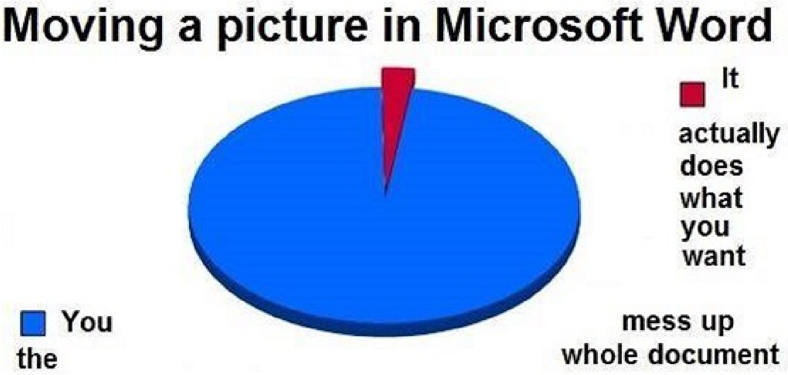
Moving a picture in Microsoft Word
It actually does what you want
You mess up the whole document

Word Processors Are Bad, M'Kay?

- Equation/figure references, bibliography, math rendering, whitespace, numbering, encoding problems, etc. [1612.04334]

With this in mind, a brute force algorithm was created to solve for the three spatial angles and semimajor/minor axes of the least-squares ellipse. However, such an algorithm was computationally intensive, slow to run, and of questionable precision. Therefore, a genetic algorithm called *TransformerEvolution* was created to solve for the same five ellipse parameters by evolving toward the minimum σ, where:

$$\sigma = \sqrt{\frac{\Sigma_{i=1}^{N}\sqrt{(x_i-x_2)^2+(y_i-y_2)^2+(z_i-z_2)^2}}{N}} \qquad (7)$$

where N is the number of position data points, (x_i, y_i, z_i) are the coordinates of a data point, and (x_2, y_2, z_2) are the coordinates of the closest point on an ellipse along the geodesic, that is, the shortest possible path, to the elliptical model. Essentially, this solves for the true least-squares ellipse by varying the parameters of potential elliptical models until the average distance from all data points on the model is minimized. This algorithm, the results of which are displayed in Fig. 7, both precisely and efficiently produces one of the ellipse equations *(5)* or *(6)* as a model of cryogenic deuterium-tritium target motion. These two similar equations are necessary because alignment along the correct axis is a required starting condition for this genetic algorithm to run

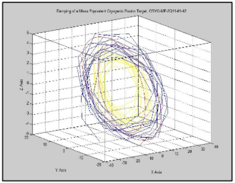

Fig. 7 3-dimensional visualization of the damping of a cryogenic target motion path
This path of motion of this target, CRYO-ME-2Q11-01-42, diminishes in size while rotating through space. Two elliptical models calculated at different points in time, depicted here in red, are accurate representations of target motion path, overlaid on the data shown in blue and yellow. Note that the magnitude of the scale on each axis varies.

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As a check also note that at late times

$$t \gg \sqrt{N}: \qquad \tilde{G}(t) = \frac{5\eta}{\pi^2}\log\frac{2}{\eta\delta}, \qquad (4.23)$$

which reproduces the plateau value from eq. (4.13).

From (4.22), we see that when $t \sim O(1)$, then $d\tilde{G}/dt \sim O(1/\sqrt{N})$, while when $t \sim O(\sqrt{N})$ at central times in the ramp, $d\tilde{G}/dt \sim 1/N$. So the plateau height will be parametrically controlled by $d\tilde{G}/dt$ at central times in the ramp, multiplied by the duration of the ramp (which is determined by the inverse of the typical gap size, which is $1/\sqrt{N}$). This gives the estimate $1/N \times \sqrt{N} \sim 1/\sqrt{N}$ for the plateau height. The more rapid growth with slope of $O(1/\sqrt{N})$ in the early part of the ramp gives a logarithmic correction, resulting in a plateau height of $\log N/\sqrt{N} \sim \eta\log(1/\eta)$.

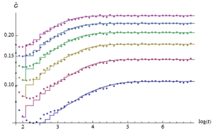

Figure 7: The estimate (4.18) for the ramp and the plateau with $\gamma = 2$ (solid lines) versus the numerically evaluated progressive time averaged regularized two point function (dots) for $\eta = 0.05, 0.075, 0.1, 0.125, 0.15, 0.175$ (from bottom to top).

4.3 Dip

In this subsection, we consider the temporal coarse graining of (4.1) with generic progressive time window of width $\Delta t = at$ (generalizing (4.14)), which we will denote by

$$\tilde{G}_a(t) = \frac{1}{at}\int_{t-at/2}^{t+at/2} dt'\,\hat{G}(t') \qquad (4.24)$$

Such a generalization does not modify the conclusions about the late part of the ramp and the plateau time. However, we do expect the precise location of the dip to be sensitive to the parameter a and therefore the specific coarse graining that we pick. As we will see, the scaling with the entropy is independent of a.

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Font, Whitespace, and General Formatting Matter a Lot!

- Ahlfors' *Complex Analysis*

We show now that $f(z)$ is identically zero in all of Ω. Let E_1 be the set on which $f(z)$ and all derivatives vanish and E_2 the set on which the function or one of the derivatives is different from zero. E_1 is open by the above reasoning, and E_2 is open because the function and all derivatives are continuous. Therefore either E_1 or E_2 must be empty. If E_2 is empty, the function is identically zero. If E_1 is empty, $f(z)$ can never vanish together with all its derivatives.

Assume that $f(z)$ is not identically zero. Then, if $f(a) = 0$, there exists a first derivative $f^{(h)}(a)$ which is different from zero. We say then that a is a *zero of order h*, and the result that we have just proved expresses that there are no zeros of infinite order. In this respect an analytic function has the same local behavior as a polynomial, and just as in the case of polynomials we find that it is possible to write $f(z) = (z - a)^h f_h(z)$ where $f_h(z)$ is analytic and $f_h(a) \neq 0$.

In the same situation, since $f_h(z)$ is continuous, $f_h(z) \neq 0$ in a neighborhood of a and $z = a$ is the only zero of $f(z)$ in this neighborhood. In other words, the zeros of an analytic function which does not vanish identically are *isolated*. This property can also be formulated as a uniqueness theorem: *If $f(z)$ and $g(z)$ are analytic in Ω, and if $f(z) = g(z)$ on a set which has an accumulation point in Ω, then $f(z)$ is identically equal to $g(z)$.* The conclusion follows by consideration of the difference $f(z) - g(z)$.

Particular instances of this result which deserve to be quoted are the following: If $f(z)$ is identically zero in a subregion of Ω, then it is identically zero in Ω, and the same is true if $f(z)$ vanishes on an arc which does not reduce to a point. We can also say that an analytic function is uniquely determined by its values on any set with an accumulation point in the region of analyticity. This does not mean that we know of any way in which the values of the function can be computed.

We consider now a function $f(z)$ which is analytic in a neighborhood of a, except perhaps at a itself. In other words, $f(z)$ shall be analytic in a region $0 < |z - a| < \delta$. The point a is called an *isolated singularity* of $f(z)$. We have already treated the case of a removable singularity. Since we can then define $f(a)$ so that $f(z)$ becomes analytic in the disk $|z - a| < \delta$, it needs no further consideration.†

If $\lim_{z \to a} f(z) = \infty$, the point a is said to be a *pole* of $f(z)$, and we set $f(a) = \infty$. There exists a $\delta' \leqq \delta$ such that $f(z) \neq 0$ for $0 < |z - a| < \delta'$. In this region the function $g(z) = 1/f(z)$ is defined and analytic. But the singularity of $g(z)$ at a is removable, and $g(z)$ has an analytic exten-

† If a is a removable singularity, $f(z)$ is frequently said to be *regular* at a; this term is sometimes used as a synonym for analytic.

sion with $g(a) = 0$. Since $g(z)$ does not vanish identically, the zero at a has a finite order, and we can write $g(z) = (z - a)^h g_h(z)$ with $g_h(a) \neq 0$. The number h is the *order* of the pole, and $f(z)$ has the representation $f(z) = (z - a)^{-h} f_h(z)$ where $f_h(z) = 1/g_h(z)$ is analytic and different from zero in a neighborhood of a. The nature of a pole is thus exactly the same as in the case of a rational function.

A function $f(z)$ which is analytic in a region Ω, except for poles, is said to be *meromorphic* in Ω. More precisely, to every $a \in \Omega$ there shall exist a neighborhood $|z - a| < \delta$, contained in Ω, such that either $f(z)$ is analytic in the whole neighborhood, or else $f(z)$ is analytic for $0 < |z - a| < \delta$, and the isolated singularity is a pole. Observe that the poles of a meromorphic function are isolated *by definition*. The quotient $f(z)/g(z)$ of two analytic functions in Ω is a meromorphic function in Ω, provided that $g(z)$ is not identically zero. The only possible poles are the zeros of $g(z)$, but a common zero of $f(z)$ and $g(z)$ can also be a removable singularity. If this is the case, the value of the quotient must be determined by continuity. More generally, the sum, the product, and the quotient of two meromorphic functions are meromorphic. The case of an identically vanishing denominator must be excluded, unless we wish to consider the constant ∞ as a meromorphic function.

For a more detailed discussion of isolated singularities, we consider the conditions (1) $\lim_{z \to a} |z - a|^\alpha |f(z)| = 0$, (2) $\lim_{z \to a} |z - a|^\alpha |f(z)| = \infty$, for real values of α. If (1) holds for a certain α, then it holds for all larger α, and hence for some integer m. Then $(z - a)^m f(z)$ has a removable singularity and vanishes for $z = a$. Either $f(z)$ is identically zero, in which case (1) holds for all α, or $(z - a)^m f(z)$ has a zero of finite order k. In the latter case it follows at once that (1) holds for all $\alpha > h = -m - k$, while (2) holds for all $\alpha < h$. Assume now that (2) holds for some α; then it holds for all smaller α, and hence for some integer n. The function $(z - a)^n f(z)$ has a pole of finite order l, and setting $h = n + l$ we find again that (1) holds for $\alpha > h$ and (2) for $\alpha < h$. The discussion shows that there are three possibilities: (i) condition (1) holds for all α, and $f(z)$ vanishes identically; (ii) there exists an integer h such that (1) holds for $\alpha > h$ and (2) for $\alpha < h$; (iii) neither (1) nor (2) holds for any α.

Case (i) is uninteresting. In case (ii) h may be called the *algebraic order* of $f(z)$ at a. It is positive in case of a pole, negative in case of a zero, and zero if $f(z)$ is analytic but $\neq 0$ at a. The remarkable thing is that the order is always an integer; there is no single-valued analytic function which tends to 0 or ∞ like a fractional power of $|z - a|$.

In the case of a pole of order h, let us apply Theorem 8 to the analytic function $(z - a)^h f(z)$. We obtain a development of the form

$$(z - a)^h f(z) = B_h + B_{h-1}(z - a) + \cdots + B_1(z - a)^{h-1} + \varphi(z)(z - a)^h$$

Font, Whitespace, and General Formatting Matter a Lot!

- Separate important equations and theorems out onto their own line
- Use (don't overuse) bold/italics for emphasis
- Font needs to be generally not painful

Even in LaTeX, Obfuscation is Common

- Rudin's *Principles of Mathematical Analysis*

We now examine this situation a little more closely. Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$. We shall show that *A contains no largest number and B contains no smallest.*

More explicitly, for every p in A we can find a rational q in A such that $p < q$, and for every p in B we can find a rational q in B such that $q < p$.

To do this, we associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{3}$$

Then

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}. \tag{4}$$

If p is in A then $p^2 - 2 < 0$, (3) shows that $q > p$, and (4) shows that $q^2 < 2$. Thus q is in A.

If p is in B then $p^2 - 2 > 0$, (3) shows that $0 < q < p$, and (4) shows that $q^2 > 2$. Thus q is in B.

Define Terms and Variables Precisely Before Using Them

Right triangles are defined by $a^2 + b^2 = c^2$. The Pythagorean Theorem says that right triangles with side lengths 3 and 4 have a hypotenuse of length 5.

Define Terms and Variables Before Using Them

Let a, b, and c be positive integers. If $a^2 + b^2 = c^2$, then a, b, and c are the side lengths of a right triangle, with c, the *hypotenuse*, being the longest. This is the *Pythagorean Theorem*. According to the Pythagorean Theorem, if 3 and 4 are two side lengths of a right triangle, then the third side has either length 5 and is the hypotenuse, or length $\sqrt{7}$, in which case the side of length 4 is the hypotenuse.

Extraneous Symbology is Obfuscation

- See [1302.5453] vs. [quant-ph/0504208]

▶ **Proposition 9.** *For a pure stabiliser state $\rho = |\psi\rangle\langle\psi|$ with associated error group $G < W$, and any $J \subset \mathcal{X}$, the entropy*

$$S(J) = S(\rho_J) = \log \frac{d_J}{|\widehat{G_J}|}. \quad (14)$$

Here, $d_J = \prod_{x \in J} d_x$ and

$$G_J \equiv \{\otimes_{x \in \mathcal{X}}\, g_x \in G : \forall x \notin J \; g_x = \mathbb{1}\} \subset G,$$

and $\widehat{G_J} = G_J / C_J$ is the quotient of G_J with respect to the center $C_J = G_J \cap C$.

In other words, $f \in S_{\hat{\alpha}}$ iff $\sigma(f)$ acts as the identity on the party α; $f \in S_\alpha$ iff $\sigma(f)$ acts as the identity on all parties $\beta \neq \alpha$. In the case $n_\alpha = 0$ we shall use a convention $S_\alpha = 0$ and $S_{\hat{\alpha}} = S$. If S is a stabilizer group of some state, we shall use the terms local (co-local) subspace and local (co-local) subgroup interchangeably.

Not All Symbology is Evil

If x is a positive real number greater than or equal to two such that no positive real number greater than or equal to one and smaller than x divides x, then we call x a *prime number*. For any positive integer n there exist two prime numbers x and y with x greater than y without loss of generality such that x minus y is at least n.

For all positive integers n, there exist two prime numbers x and y, $x > y$, such that $x - y \geq n$.

Adjust Level of Formality to the Audience

Definition (Refinement)

A *refinement* of a set X is a family of sets P such that:

- $\emptyset \notin P$
- $\bigcup_{A \in P} A = X$
- $(\forall A, B \in P)\, A \neq B \implies A \cap B = \emptyset$

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- Wat?!

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- Wat?!
- I took this from Wikipedia!

Adjust Level of Formality to the Audience

Definition (Refinement)

A *refinement* of a set X is a way of breaking apart the set X into different pieces.

. . . But Still Be Precise

Definition (Refinement)

A *refinement* of a set X is a way of breaking apart the set X into a collection of nonempty sets, such that the sets don't overlap and that all of the sets taken together make up all of X.

Choose Language Appropriately to Match Intuition

Definition (Partition)

A *partition* of a set X is a way of breaking apart the set X into a collection of nonempty sets, such that the sets don't overlap and that all of the sets taken together make up all of X.

An Example is Worth a Thousand Words

Definition (Partition)

A *partition* of a set X is a way of breaking apart the set X into a collection of nonempty sets, such that the sets don't overlap and that all of the sets taken together make up all of X.

- The set {1,2,3,4,5,6,7} can be partitioned into the subsets

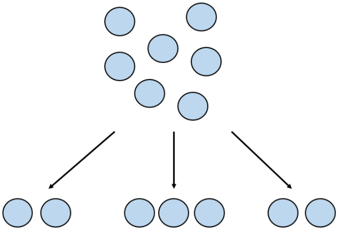
$$\{1, 2\}, \{3, 4, 5\}, \{6, 7\}.$$

- This is also written as $12|345|67$.

A Picture is Worth a Thousand Words

Definition (Partition)

A *partition* of a set X is a way of breaking apart the set X into a collection of nonempty sets, such that the sets don't overlap and that all of the sets taken together make up all of X.



Use Multiple Modes of Expression to Reach Wider Audiences

Definition (Partition)

A *partition* of a set X is a way of breaking apart the set X into a collection of nonempty sets, such that the sets don't overlap and that all of the sets taken together make up all of X.

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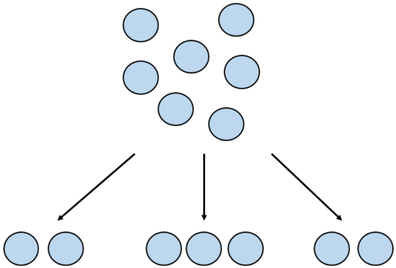

Overview

We'll Use ShareLaTeX to Avoid Installation

- https://www.sharelatex.com (Like Google Docs but for LaTeX)
- (Or your local LaTeX installation)

Style and Format are Flexible and Widely Varied

- See example documents

10 Minute Break

- Go to the bathroom, eat a snack, text, (s)nap, chat, whatever
- (We'll have this every day)

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Slide Titles Should Concisely Drive the Point Home

- Typical audience not going to remember intricate details
- Average number N of audience members that can write down your formulas from memory after:

$$N = P : \mathbb{R}^0 \to \mathcal{B}^{a,b} \cap \mathcal{L}^y(n_0 n^e!)$$

- 46.53% of the numbers on this slide either made up or blend into all other numbers
- Here's another gratuitous formula (POTATOES):

$$\chi = \prod_{i=0}^{n-1} \int_{-\infty}^{\infty} \frac{dx^i}{(2\pi\hbar c^3)^{2/5}} e^{-x_i^2/x_0^2} \Gamma(x_i) \dim \bigoplus_{j=0}^{i} H^j(T^n, \mathbb{Z})$$

- Too many bullet points clutters visually and distracts from the point
- Seriously, use multiple slides instead of doing this. One idea per slide!

Keep Your Slides Clean

- Too many bullet points clutters visually and distracts from the point

Slide Titles Should Concisely Drive the Point Home

- Typical audience not going to remember intricate details

Minimize Formula Use; Interpret the Ones You Have

- Average number N of audience members that can write down your formulas from memory after:

$$N = P : \mathbb{R}^0 \to \mathcal{B}^{a,b} \cap \mathcal{L}^y(n_0 n^e!)$$

- Not a large number!

Reveal Information Only as You Need It

- Sometimes, it is necessary or useful to have a bit more information in a slide

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- Don't have to present it all at once, though

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- Don't have to present it all at once, though
- Wait so that audience is not distracted by all the info

Reveal Information Only as You Need It

- Sometimes, it is necessary or useful to have a bit more information in a slide
- Don't have to present it all at once, though
- Wait so that audience is not distracted by all the info
- Give them time to digest what you want them to point by point

Minimize Formula Use; Interpret the Ones You Have

- Who can write down that second formula from five slides ago?

Minimize Formula Use; Interpret the Ones You Have

- Who can write down that second formula from five slides ago?
- Who can remember what I said was important to remember about it?

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Back to ShareLaTeX!

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Style and Format are Flexible and Widely Varied

- See example documents

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Interlude - Theorems and How to Prove Them

- Time for board work / theorem-proof worksheet

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Presentation Details

- See syllabus for suggested topics - pick one or find one, as a group
- Wed. and Thurs. for research, presenting Friday
- 10-15 min including questions
- Tell a complete story
- Consult me for help!

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Contact Me Whenever You Want

- Email: mdecross@sas.upenn.edu - most reliable
- Office: DRL 2N3D - less reliable