Penn Summer Prep Program
Explorations in Mathematical Inquiry - PSPR-008-920
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Module Description:

What exactly does a research mathematician do all day? A popular image of a mathematician is that of a coffee-addicted lone wolf, working late into the night in an isolated office on the next fantastically complicated breakthrough idea that no other living soul will read. We will show that this couldn't be further from the truth: modern math research is *collaborative*, *incremental*, and *grounded in natural and reasonable questions*. Furthermore, many of the most famous mathematicians (and other scientists) are well-known not solely due to their breakthrough work but more importantly the remarkable clarity and precision with which they communicate their ideas. In this module, students will have the opportunity to play in a mathematical research sandbox, investigating small-scale open-ended scenarios and reviewing one major modern research idea in teams. We will explain how to present the mathematical progress we make by typesetting in LaTeX and Beamer, and emphasize how to use the advantages offered by these tools to improve technical communication.

Module Objectives: Via this course, students will:

- Understand the process of mathematical inquiry as experienced by professional mathematicians.
- Be exposed to several modern research-level topics in mathematics.
- Become proficient in distilling relevant and irrelevant information in reviewing the state-of-the-art on a new topic of investigation.
- Gain technical facility in typesetting mathematical and scientific work in LaTeX and Beamer.
- Develop techniques for communicating technical concepts more effectively both verbally and in writing

Assignments: In sessions 2-6, students will work in groups of 2-4 during the class time to investigate an open-ended, appropriately level-adjusted mathematical research problem. One group will present their progress on a chalkboard at the end of each session. Groups will produce a short (~2 page) writeup, typeset in LaTeX, of their progress on one particular problem from the first week. This will be due by Session 6, to be returned with feedback from the instructor at the beginning of the next session. During sessions 7 and 8, student groups will work to produce a presentation in Beamer reviewing a known but difficult and interesting topic, to be presented with immediate feedback during session 9. Shorter (~15-20 minute) tasks involving development of mathematical communication skills will be assigned daily. **Students will need to bring a laptop or otherwise obtain computer access since the assignments both in and out of class will require software use.**

Schedule:

Session 1: Introduction to Mathematical Communication

- Introduction and overview of the module
- Discussion of schedule and format for following sessions
- Discussion of student projects and presentations
- Introductory survey
- Mathematical communication techniques
- Introduction to typesetting in LaTeX
- Introduction to presentation design in Beamer
- Introduction to theorems and proofs

Assignments: Finish Theorem/Proof for Session 3, Introduction to LaTeX and Beamer Bootcamp for Session 2

Session 2: Stacking Cards

A single playing card can be extended up to half its length over the edge of a table before falling off. Given two playing cards, however, by shifting the bottom one backwards, the top card can extend more than half the length of one card over the edge of a table. What's the maximum length an entire deck of playing cards can be extended over the edge of a table without any cards falling off? What if you have as many cards as you want – is there a limit to how far you can extend the cards? (If so, what is it?) What if you can put multiple cards at the same level in the stack?

- Center of mass: a lightning introduction (or review)
- Open work time with instructor assistance
- Discussion of class approaches to the problem
- Presentation of one group's progress on the problem

Assignment: Finish Theorem/Proof for Session 3

Session 3: Random Walks and Cliffs

A blind man standing one step from the edge of a cliff walks randomly either towards or away from the cliff, with probability ½ of "towards" and probability ½ of "away." If he keeps walking randomly forever, what is the probability that he eventually falls off the cliff at some point? What if the probability is p that he steps towards the cliff? What if n steps away from the cliff there is a helicopter that will take him to safety permanently? What if n steps away from the cliff there is another cliff?

- Probability and Markov chains: a lightning introduction (or review)
- Open work time with instructor assistance
- Discussion of class approaches to the problem

- Presentation of one group's progress on the problem

Assignment: Avoiding Symbology

Session 4: Parking Spaces

A street has some number of parking spaces along the sidewalk, each a single car length long. The parking spaces start empty, then fill up as cars drive up and park in the remaining open spaces. However, each car, when parking, must follow the rule that they leave a space in between them and the nearest car, so that they don't box people out. If each car chooses uniformly randomly among the open parking spaces that respect this rule, what fraction of the parking spaces are full once no more permissible spaces are available? What does this fraction tend to as the street gets longer and longer? What if the cars are biased towards certain spots over others? What if we get rid of the parking space boundaries and the rule about adjacent cars?

- Recursive sequences and limits: a lightning introduction (or review)
- Open work time with instructor assistance
- Discussion of class approaches to the problem
- Presentation of one group's progress on the problem

Assignment: Complete problem writeup for Session 6

Session 5: Tower of Hanoi

The "Tower of Hanoi" is a game involving a series of pegs and a stack of disks of various sizes with holes in their centers for the pegs. The disks are arranged on one peg in a pyramidal shape so that the largest is at the bottom and smallest at the top. The objective is to transfer the pyramid from one peg to another in the minimal number of moves, but there's a catch: no disk may be placed (directly or indirectly) above a disk that's smaller than itself at any point. If there are three pegs and five disks, what is the minimal number of moves? What is the maximal number of moves? How many configurations of disks on pegs are possible? What is the minimal number of moves of the kth largest disk? What if there are an arbitrary number of disks? What if the three pegs are arranged in a triangle and disks may only be moved clockwise? What if the disks are magnets that are flipped each time they are moved, and only sides of opposite polarity may touch? What if there are four (or more) pegs?

- Games and graphs: a lightning introduction (or review)
- Open work time with instructor assistance
- Discussion of class approaches to the problem
- Presentation of one group's progress on the problem

Assignment: Complete problem writeup by the next day (Session 6).

Session 6: Multipoles

Two electric charges of opposite sign placed near each other form an *electric dipole,* where the electric field falls off as 1/r^3 instead of 1/r^2 along a line far away from the charges. What configurations of multiple electric charges along a line yield electric 2^n –pole configurations, defined as those where the field falls off as 1/r^(n+2) along a line far away from the charges? What if the charges are restricted to be evenly spaced? Do you find any interesting identities you can prove? What if Coulomb's law was modified to 1/r^(2+x) for some x? Can you find other interesting variations to explore?

- Coulomb's law and binomial expansions: a lightning introduction (or review)
- Open work time with instructor assistance
- Discussion of class approaches to the problem
- Presentation of one group's progress on the problem

Assignment: ArXiv Search and Summary

Session 7: Student Project Work Day #1

- Discussion of presentation day logistics
- Open work time with instructor assistance
- Open class discussion of ongoing mathematical, technical, and communication challenges

Assignment: Work on presentation

Session 8: Student Project Work Day #2

- Open work time with instructor assistance
- Open class discussion of ongoing mathematical, technical, and communication challenges

Assignment: Work on presentation

Session 9: Student Project Presentations

- Group and instructor presentations and questions
- Presentation feedback and concluding remarks

Assignment: Complete evaluation

Suggested Group Presentation Topics (* indicates calculus likely required):
Volumes of balls and areas of spheres in arbitrary dimension*
Non-orientable surfaces
The Euler characteristic and Betti numbers
Vertex colorings and the chromatic number of graphs
Four-color theorem and Headwood conjecture
Gauss linking and writhe of knots

Prime numbers and the Riemann zeta function*
The modular group and its action on the torus
The Doomsday algorithm
Hall’s marriage theorem
Fractal dimensions
The Cantor set
Weierstrass functions*
Cardinality and Cantor’s diagonal argument
Peano curves
Fermat’s Little Theorem
Non-Euclidean geometry
Bernoulli numbers*
Martingales and the optional stopping theorem
Crystallographic restriction
Self-referencing sequences
Applications of rings and ideals
Homotopy groups
The Hopf fibration
The game of Nim (and variants)
…or your choice!